# Interaction Laws of Monads and Comonads 

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#### Abstract

We introduce and study functor-functor and monad-comonad interaction laws as mathematical objects to describe interaction of effectful computations with behaviors of effect-performing machines. Monad-comonad interaction laws are monoid objects of the monoidal category of functorfunctor interaction laws. We show that, for suitable generalizations of the concepts of dual and Sweedler dual, the greatest functor resp. monad interacting with a given functor or comonad is its dual while the greatest comonad interacting with a given monad is its Sweedler dual. We relate monad-comonad interaction laws to stateful runners. We show that functor-functor interaction laws are Chu spaces over the category of endofunctors taken with the Day convolution monoidal structure. Hasegawa's glueing endows the category of these Chu spaces with a monoidal structure whose monoid objects are monad-comonad interaction laws.


## 1 Introduction

What does it mean to run an effectful program, abstracted into a computation?
In this paper, we take the view that an effectful computation does not perform its effects; those are to be provided externally. The computation can only proceed if placed in an environment that can provide its effects, e.g, respond to the computation's requests for input, listen to its output, resolve its nondeterministic choices by tossing a coin, consistently respond to its fetch and store commands. Abstractly, such an environment is a machine whose implementation is opaque to us; we can witness its behavior, its evolution through externally visible states.

To formalize this intuition, we follow Moggi [23] and Plotkin and Power [26] in regards to allowed computations (the chosen notions of computation) and describe them using a monad (resp. algebraic theory) $T$ on the category of types and functions that we want to compute on. Allowed machine behaviors (the chosen notion of machine behavior), at the same time, are described with a comonad $D$. An operational semantics is then described by what we call an interaction law, a natural transformation $\psi: T X \times D Y \rightarrow X \times Y$ compatible with the (co)unit and (co)multiplication. This polymorphic function sends a computation $(T X)$ and a machine behavior from some initial state ( $D Y$ ) into a return value $X$ and a final state $Y$. It is also fine to work with notions of computation and machine behavior that do not include "just returning" or/and are not closed under sequential composition; those can be described with plain functors instead of a monad and a comonad.

We take special interest in the questions (a) which is the "greatest" comonad interacting with the given monad $T$ (so any interaction law of $T$ with any comonad would factor through the canonical interaction law of $T$ with this comonad)? and (b) which is the "greatest" monad (resp. functor) interacting with a given comonad $D$ (or functor $G$ )? To answer these, we draw inspiration from algebra, where the dual of a vector space $V$ is $V^{\circ}=V \rightarrow \mathbb{K}$. The answer to (b) turns out to be: the dual of $D$ (resp. $G$ ), under a suitably generalized concept of dual. Question (a) is harder. To answer it, we need to generalize the concept of what is called the Sweedler dual. The greatest comonad interacting with $T$ is the Sweedler dual of $T$.

The contributions in this paper are the following:
(i) We introduce functor-functor interaction laws, define the dual of a functor, and show that the greatest functor interacting with a given functor is its dual (Section 22).
(ii) We study monad-comonad interaction laws as monoid objects of the category of functor-functor interaction laws. We show that the dual lifts from functors to comonads and that the greatest monad interacting with a given comonad is its dual whereas for monads it does not lift like this; for the greatest comonad interacting with a monad, the Sweedler dual is needed (Section 3).
(iii) We relate monad-comonad interaction laws to stateful runners of Uustalu 36] (Section 44).
(iv) Using the Day convolution and duoidal categories, we recast monad-comonad interaction laws as monoid-comonoid interaction laws, and relate them to two standard constructions: Chu spaces and Hasegawa's glueing (Section 6). This gives us a method for computing the Sweedler duals of free monoids (monads) and their quotients by equations.
We also introduce and study residual functor-functor interaction laws, monad-comonad interaction laws and stateful runners as generalizations where the machine need not be able to perform all effects of the computation (Section 5).

We assume the reader to be familiar with adjunctions/monads/comonads, extensive categories [11], Cartesian closed categories, ends/coends (the end-coend calculus 920). In a nutshell, extensive categories are categories with well-behaved finite coproducts.

Throughout most of the paper (Sections $2 \sqrt{4}$ ), we work with one fixed base category $\mathcal{C}$ that we assume to be extensive with finite products. For some constructions (the dual of a functor), we also need that $\mathcal{C}$ is Cartesian closed. For the same constructions, we also use certain ends that we either explicitly show to exist or only use when they happen to exist. We also rely on Cartesian closedness in most examples.

## 2 Functor-functor interaction

We begin with functor-functor interaction, to then proceed to the monad-comonad interaction laws in the next section.

### 2.1 Functor-functor interaction laws

In a functor-functor interaction law, computations over a set of values $X$ are elements of $F X$ where $F$ is a given functor. Machine behaviors over a set of states $Y$ are elements of $G Y$ where $G$ is another given functor. Any allowed computation and any allowed machine behavior can help each other reach a return value and a final state by interacting as prescribed.

We define an functor-functor interaction law on $\mathcal{C}$ to be given by two endofunctors $F, G$ together with a family of maps

$$
\phi_{X, Y}: F X \times G Y \rightarrow X \times Y
$$

natural in $X$ and $Y$.
Example 1. The archetypical example of a functor-functor interaction law is defined by $F X=A \Rightarrow X$, $G Y=A \times Y$, and $\phi(f,(a, y))=(f a, y)$ for some fixed object $A$. But we can also take, e.g., $F X=A \Rightarrow X$, $G Y=C \times Y$, and $\phi(f,(c, y))=(f(h c), y)$ for some fixed map $h: C \rightarrow A$.

Example 2. A more interesting example is obtained by taking $F X=A \Rightarrow(B \times X), G Y=A \times(B \Rightarrow Y)$, $\phi(f,(a, g))=$ let $(b, x) \leftarrow f a$ in $(x, g b)$. We can vary this by taking $G Y=(A \Rightarrow B) \Rightarrow(A \times Y)$ and $\phi(f, h)=$ let $\left\langle f_{0}, f_{1}\right\rangle \leftarrow f ;(a, y) \leftarrow h f_{0}$ in $\left(f_{1} a, y\right)$.
Example 3. If $\mathcal{C}$ has the relevant initial algebras and final coalgebras, we can get interaction laws by iterating the above interactions, e.g., with $F X=\mu Z . X+(A \Rightarrow(B \times Z))$ and $G Y=\nu W . Y \times(A \times(B \Rightarrow$ $W)$ ), or with $F X=\nu Z . X+(A \Rightarrow(B \times Z))$ and $G Y=\mu W . Y \times(A \times(B \Rightarrow W))$. We will shortly explain the construction of $\phi$ in the first of these two cases.

An interaction law map between $(F, G, \phi),\left(F^{\prime}, G^{\prime}, \phi^{\prime}\right)$ is given by natural transformations $f: F \rightarrow F^{\prime}$, $g: G^{\prime} \rightarrow G$ such that $\phi_{X, Y} \circ\left(\operatorname{id}_{F X} \times g_{Y}\right)=\phi_{X, Y}^{\prime} \circ\left(f_{X} \times \mathrm{id}_{G^{\prime} Y}\right)$.

Interaction laws form a category $\mathbf{I L}(\mathcal{C})$, where the identity on $(F, G, \phi)$ is $\left(\mathrm{id}_{F}, \mathrm{id}_{G}\right)$, and the composition of $(f, g):(F, G, \phi) \rightarrow\left(F^{\prime}, G^{\prime}, \phi^{\prime}\right)$ and $\left(f^{\prime}, g^{\prime}\right):\left(F^{\prime}, G^{\prime}, \phi^{\prime}\right) \rightarrow\left(F^{\prime \prime}, G^{\prime \prime}, \phi^{\prime \prime}\right)$ is $\left(f^{\prime} \circ f, g \circ g^{\prime}\right)$. The condition on a interaction law map is met for the composition because of the commutation of the diagram


The composition monoidal structure of $[\mathcal{C}, \mathcal{C}]$ induces a similar monoidal structure on the category $\mathbf{I L}(\mathcal{C})$. The tensorial unit is (Id, Id, $\left.\mathrm{id}_{\mathrm{Id} \times \mathrm{Id}}\right)$. The tensor of $(F, G, \phi)$ and $(J, K, \psi)$ is $(F \cdot J, G \cdot K, \psi \circ \phi \cdot(J \times K))$. The tensor of $(f, g):(F, G, \phi) \rightarrow\left(F^{\prime}, G^{\prime}, \phi^{\prime}\right)$ and $(j, k):(J, K, \psi) \rightarrow\left(J^{\prime}, K^{\prime}, \psi^{\prime}\right)$ is $(f \cdot j, g \cdot k)$. The condition on an interaction law map is met by the commutation of


### 2.2 Two degeneracy results

Here are two simple degeneracy results. We first recall the notion of operation for monads and functors.

A comment on operations The concept of (algebraic) operation of a monad can be defined in several ways. Given a monad $T$, an $n$-ary operation of $T$ can be defined to be a natural transformation $c^{\prime}:(T X)^{n} \rightarrow T X$ (where $X^{n}$ is $n$-fold product of $X$ with itself) satisfying


This is the format used by Plotkin and Power [27. (We do not require here that $T$ is strong and drop compatibility with the strength.) Alternatively, we can say that it is a natural transformation $c: X^{n} \rightarrow$ $T X$ and drop the requirement of commutation with $\mu$, as done by Jaskelioff and Moggi [19].

We can also say that it is a map $1 \rightarrow T n$ (a "generic effect" in the sense of Plotkin and Power [26]) but, for this to amount to the same as the previous alternative, one needs that $T$ is strong.

If a finitary set monad $T$ is determined by a Lawvere theory $(\mathcal{L}, L)$ where $\mathcal{L}$ is a category with finite products and $L: \mathbb{F}^{\text {op }} \rightarrow \mathcal{L}$ is identity on objects and strictly product-preserving, one can say that an operation is a map $n \rightarrow 1$ in $\mathcal{L}$. Given a monad $T$ on an arbitrary category $\mathcal{C}$, its large Lawvere theory is $\left((\mathbf{K l}(T))^{\mathrm{op}}, J^{\mathrm{op}}\right)$ where $J: \mathcal{C} \rightarrow \mathbf{K l}(T)$ is the left adjoint of the Kleisli adjunction. A map $n \rightarrow 1$ in $(\mathbf{K l}(T))^{\mathrm{op}}$ is the same as a map $1 \rightarrow T n$ in $\mathcal{C}$.

In this paper, we prefer to work with operations as maps $c: X^{n} \rightarrow T X$ because this format is intuitive and economic in proofs by diagram chasing but also because it makes sense when $T$ is only a functor and not a monad.

Functors with a nullary operation For the functor Maybe $X=($ just : $X)+($ nothing : 1), it should be clear intuitively that it cannot have a nondegenerate interacting functor: from the element nothing ${ }_{0}$ of Maybe 0, one cannot possibly extract an element of 0 . Formally, we have the following theorem.

Theorem 1. If a functor $F$ has a nullary operation, i.e., comes with a family of maps $c_{X}: 1 \rightarrow F X$ natural in $X$, then any interacting functor $G$ is constant zero, i.e., $G Y \cong 0$ for any $Y$.

Proof. Indeed, for any $Y$, we have the map

$$
G Y \xrightarrow{\langle!, \text { id }\rangle} 1 \times G Y \xrightarrow{c_{0} \times \text { id }} F 0 \times G Y \xrightarrow{\phi_{0, Y}} 0 \times Y \xrightarrow{\text { fst }} 0
$$

Since the initial object of an extensive category is strict (any map to 0 is an isomorphism), we can conclude that $G Y \cong 0$.

The theorem applies to Maybe since it comes with a nullary operation nothing ${ }_{X}: 1 \rightarrow$ Maybe $X$.

Functors with a commutative binary operation A similar no-go theorem holds for commutative binary operations.

Theorem 2. If a functor $F$ has a commutative binary operation, i.e., comes with a family of maps $c_{X}: X \times X \rightarrow F X$ natural in $X$ such that $c_{X}=c_{X} \circ \operatorname{sym}_{X, X}$, then any interacting functor $G$ is constant zero, i.e., $G Y \cong 0$ for any $Y$.

Proof. Let $\mathbb{B}=(\mathrm{tt}: 1)+(\mathrm{ff}: 1)$. Then, for any $Y$, the map

$$
f_{Y}=G Y \xrightarrow{\langle!, \mathrm{id}\rangle} 1 \times G Y \xrightarrow{\langle\mathrm{tt}, \mathrm{ff}\rangle}(\mathbb{B} \times \mathbb{B}) \times G Y \xrightarrow{c_{\mathbb{B}} \times \mathrm{id}} F \mathbb{B} \times G Y \xrightarrow{\theta_{\mathbb{B}, Y}} \mathbb{B} \times Y \xrightarrow{\mathrm{fst}} \mathbb{B}
$$

has the property that not $\circ f_{Y}=f_{Y}$ :


By stability of coproducts under pullback in an extensive category, we can pull the coprojections of $\mathbb{B}$ back along $f_{Y}$

and the result is a pullback again.
Now we have

so by disjointness of coproducts in an extensive category we have a map $h_{Y}^{\prime}: P Y \rightarrow 0$ as a unique map into the pullback 0 of $t t$ and $f$ :


Similarly we get a map $k_{Y}^{\prime}: Q Y \rightarrow 0$. Hence we have a map $f_{Y}^{\prime}: G Y \rightarrow 0$ from copairing $h_{Y}^{\prime}$ and $k_{Y}^{\prime}$ :


Since the initial object is strict in an extensive category, it follows that $G Y \cong 0$.

The degeneracy problem can be overcome by switching to a residual version of interaction laws, discussed in detail in Section 5 below. As a sneak preview, given a monad $R$ on $\mathcal{C}$, an $R$-residual functorfunctor interaction law is given by two endofunctors $F, G$ and a family of maps $\phi: F X \times G Y \rightarrow R(X \times Y)$ natural in $X, Y$. The monoidal structure of the category $\mathbf{I L}(\mathcal{C}, R)$ of $R$-residual functor-functor interaction laws relies on the monad structure of $R$. Typically, one would use the maybe, finite nonempty multiset or finite multiset monad as $R$.

### 2.3 On the structure of $\operatorname{IL}(\mathcal{C})$

We now look at some ways to construct functor-functor interaction laws systematically.
"Stretching" Given a functor-functor interaction law $(F, G, \phi)$ and natural transformations $f: F^{\prime} \rightarrow F$ and $g: G^{\prime} \rightarrow G$, we have a functor-functor interaction law $\left(F^{\prime}, G^{\prime}, \phi \circ(f \times g)\right.$ ).

Self-duality For any functor-functor interaction law $(F, G, \phi)$, we have another functor-functor interaction law $(F, G, \phi)^{\text {rev }}=\left(G, F, \phi^{\mathrm{rev}}\right)$ where $\phi_{X, Y}^{\mathrm{rev}}=\operatorname{sym}_{Y, X} \circ \phi_{Y, X} \circ \operatorname{sym}_{F X, G Y}$. This object mapping extends to maps by $(f, g)^{\mathrm{rev}}=(g, f)$, so we have a functor $(-)^{\mathrm{rev}}:(\mathbf{I L}(\mathcal{C}))^{\mathrm{op}} \rightarrow \mathbf{I L}(\mathcal{C})$. The functor $(-)^{\mathrm{rev}}$ is an isomorphism between $(\mathbf{I L}(\mathcal{C}))^{\text {op }}$ and $\mathbf{I L}(\mathcal{C})$.

The final functor-functor interaction law The final functor-functor interaction law is $(1,0, \phi)$ where $\phi_{X, Y}=1 \times 0 \xrightarrow{\text { snd }} 0 \xrightarrow{?} X \times Y$. By self-duality, the initial functor-functor interaction law is $\left(0,1, \phi^{\text {rev }}\right)$.

Product of two functor-functor interaction laws Given two functor-functor interaction laws $\left(F_{0}, G_{0}, \phi_{0}\right)$ and $\left(F_{1}, G_{1}, \phi_{1}\right)$, their product is $\left(F_{0} \times F_{1}, G_{0}+G_{1}, \phi\right)$ where

$$
\begin{aligned}
\phi_{X, Y}=( & \left.F_{0} X \times F_{1} X\right) \times\left(G_{0} Y+G_{1} Y\right) \xrightarrow{\text { rdist }} \\
& \left(F_{0} X \times F_{1} X\right) \times G_{0} Y+\left(F_{0} X \times F_{1} X\right) \times G_{1} Y \xrightarrow{\text { fst } \times \text { id }+ \text { snd } \times \text { id }} \\
& F_{0} X \times G_{0} Y+F_{1} X \times G_{1} Y \xrightarrow{\phi_{0 X, Y}+\phi_{1 X, Y}} X \times Y+X \times Y \xrightarrow{\nabla} X \times Y
\end{aligned}
$$

By self-duality, the coproduct of $\left(G_{0}, F_{0}, \phi_{0}^{\mathrm{rev}}\right)$ and $\left(G_{1}, F_{1}, \phi_{1}^{\mathrm{rev}}\right)$ is $\left(G_{0}+G_{1}, F_{0} \times F_{1}, \phi^{\mathrm{rev}}\right)$.
An initial algebra-final coalgebra construction Assume that $\mathcal{C}$ has the relevant initial algebras and final coalgebras. Given functors $F, G: \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$ and a family of maps $\phi_{X, Y, W, Z}: F(X, Z) \times G(Y, W) \rightarrow X \times$ $Y+Z \times W$ natural in $X, Y, Z, W$. Then we have an interaction law ( $F^{\prime}, G^{\prime}, \phi^{\prime}$ ) where $F^{\prime} X=\mu Z . F(X, Z)$, $G^{\prime} X=\nu W . G(Y, W)$ and $\phi^{\prime}$ is constructed as follows. We equip $G^{\prime} Y \Rightarrow(X \times Y)$ with an $F(X,-)$-algebra structure $\theta_{X, Y}^{0}$ by currying the map

$$
\begin{aligned}
& \theta_{X, Y}=F\left(X, G^{\prime} Y \Rightarrow(X \times Y)\right) \times G^{\prime} Y \xrightarrow{{\text { id } \times \text { out }_{G(Y,-)}}} \\
& F\left(X, G^{\prime} Y \Rightarrow(X \times Y)\right) \times G\left(Y, G^{\prime} Y\right) \xrightarrow{\phi_{X, Y, G^{\prime} Y \Rightarrow(X \times Y), G^{\prime} Y}} \\
& X \times Y+\left(G^{\prime} Y \Rightarrow(X \times Y)\right) \times G^{\prime} Y \xrightarrow{\text { id+ } \mathrm{ev}} X \times Y+X \times Y \xrightarrow{\nabla} X+Y
\end{aligned}
$$

The map $\phi_{X, Y}^{\prime}$ is obtained by uncurrying the corresponding unique map $\phi_{X, Y}^{0}: F^{\prime} X \rightarrow G^{\prime} Y \Rightarrow(X \times Y)$ from the initial $F(X,-)$-algebra.

Restricting to fixed $F$ or $G$ Sometimes it is of interest to focus on interaction laws of a fixed first functor $F$ or a fixed second functor $G$ (and accordingly on interaction law maps with the first resp. the second natural transformation the identity natural transformation on $F$ resp. $G$ ). We denote the corresponding categories by $\left.\mathbf{I L}(\mathcal{C})\right|_{F,-}$ and $\left.\mathbf{I L}(\mathcal{C})\right|_{-, G}$. The isomorphism of categories $\mathbf{I L}(\mathcal{C})^{\text {op }} \cong \mathbf{I L}(\mathcal{C})$ given by $(-)^{\text {rev }}$ restricts to $\left.\left(\left.\mathbf{I L}(\mathcal{C})\right|_{F,-}\right)^{\mathrm{op}} \cong \mathbf{I L}(\mathcal{C})\right|_{-, F}$.

The final object of $\left.\mathbf{I L}(\mathcal{C})\right|_{F,-}$ is $(F, 0, \phi)$ where

$$
\phi_{X, Y}=F X \times 0 \xrightarrow{\text { snd }} 0 \xrightarrow{?} X \times Y
$$

By self-duality, the initial object of $\left.\mathbf{I L}(\mathcal{C})\right|_{-, F}$ is $\left(0, F, \phi^{\text {rev }}\right)$. About the initial object of $\left.\mathbf{I L}(\mathcal{C})\right|_{F,-}$ we will see in the next subsection.

### 2.4 Functor-functor interaction in terms of the dual

If $\mathcal{C}$ is Cartesian closed, then we can define the dual $G^{\circ}$ of an endofunctor $G$ on $\mathcal{C}$ by

$$
G^{\circ} X=\int_{Y} G Y \Rightarrow(X \times Y)
$$

provided that this end exists, and the dual $g^{\circ}: G^{\circ} \rightarrow G^{\prime \circ}$ of a natural transformation $g: G^{\prime} \rightarrow G$ by

$$
g_{X}^{\circ}=\int_{Y} g_{Y} \Rightarrow(X \times Y)
$$

This construction is contravariantly functorial, i.e., if the dual is everywhere defined, then we have $(-)^{\circ}$ : $[\mathcal{C}, \mathcal{C}]^{\text {op }} \rightarrow[\mathcal{C}, \mathcal{C}]$. The existence of all the ends required for this is a strong condition (e.g., a small category that has all limits under classical logic is necessarily a preorder by an argument by Freyd [21). But for well-definedness and functoriality of $(-)^{\circ}$ in the general case, it suffices to restrict it to those endofunctors on $\mathcal{C}$ that happen to have the dual or, if one so wishes, to some well-delineated smaller class of functors that are guaranteed to have it (e.g., to finitary functors if $\mathcal{C}$ is locally finitely presentable). For $(-)^{\circ}$ to be a contravariant endofunctor on some full subcategory of $[\mathcal{C}, \mathcal{C}]$, we can restrict it to those endofunctors on $\mathcal{C}$ that are dualizable any finite number of times or to some other class of functors closed under the dual. Throughout this paper, we deliberately ignore this existence issue: we either explicitly prove for the ends of interest that they exist or we use such ends on the assumption that they happen to exist.

We have

$$
\begin{aligned}
& \int_{Y} \mathcal{C}(G Y, \overbrace{\int_{X} F X \Rightarrow(Y \times X)}^{F^{\circ} Y}) \cong \int_{Y, X} \mathcal{C}(G Y \times F X, Y \times X) \\
& \cong \int_{X, Y} \mathcal{C}(F X \times G Y, X \times Y) \cong \int_{X} \mathcal{C}(F X, \underbrace{\int_{Y} G Y \Rightarrow(X \times Y)}_{G^{\circ} X})
\end{aligned}
$$

where by the top-level ends we just indicate collections (not necessarily sets) of natural transformations, so existence is not an issue.

Thus, to have a functor-functor interaction law of $F, G$ is the same as to have a natural transformation $\phi: F \rightarrow G^{\circ}$ or a natural transformation $\phi: G \rightarrow F^{\circ}$.

Under the first of these identifications, an interaction law map between ( $F, G, \phi$ ) and ( $F^{\prime}, G^{\prime}, \phi^{\prime}$ ) is given by natural transformations $f: F \rightarrow F^{\prime}$ and $g: G^{\prime} \rightarrow G$ satisfying $g^{\circ} \circ \phi=\phi^{\prime} \circ f$. Under the second one, an interaction law map between $(F, G, \phi)$ and $\left(F^{\prime}, G^{\prime}, \phi^{\prime}\right)$ is given by natural transformations $f: F \rightarrow F^{\prime}$ and $g: G^{\prime} \rightarrow G$ satisfying $\phi \circ g=f^{\circ} \circ \phi^{\prime}$.

We have thus established that these categories are isomorphic:
(o) the category $\mathbf{I L}(\mathcal{C})$ of functor-functor interaction laws;
(i) the comma category $[\mathcal{C}, \mathcal{C}] \downarrow(-)^{\circ}$ of triples of two functors $F, G$ and a natural transformation $F \rightarrow G^{\circ}$;
(ii) the comma category $(-)^{\mathrm{oop}} \downarrow[\mathcal{C}, \mathcal{C}]^{\mathrm{op}}$ of triples of two functors $F, G$ and a natural transformation $G \rightarrow F^{\circ}$.

From these observations it is immediate that $\left.\mathbf{I L}(\mathcal{C})\right|_{-, G} \cong[\mathcal{C}, \mathcal{C}] / G^{\circ}$ and $\left.\mathbf{I L}(\mathcal{C})\right|_{F,-} \cong F^{\circ} \backslash[\mathcal{C}, \mathcal{C}]^{\mathrm{op}}$. Hence, the initial object of $\left.\mathbf{I L}(\mathcal{C})\right|_{-, G}$ is $(0, G, \ldots)$ while the final object is $\left(G^{\circ}, G, \ldots\right)$. The initial object of $\left.\mathbf{I L}(\mathcal{C})\right|_{F,-}$ is $\left(F, F^{\circ}, \ldots\right)$ while the final object is $(F, 0, \ldots)$.

### 2.5 Dual for some constructions on functors

Here are constructions of the dual for some basic constructions of functors.
Dual of the identity functor $\mathrm{Id}^{\circ} \cong \mathrm{Id}$.
Proof. Let $G Y=Y$. Then

$$
\begin{aligned}
G^{\circ} X & =\int_{Y} Y \Rightarrow(X \times Y) \\
& \cong \int_{Y}(1 \Rightarrow Y) \Rightarrow(X \times Y) \\
& \cong X \times 1 \\
& \cong X
\end{aligned}
$$

Duals of terminal functor, products of a functor, initial functor, coproduct of two functors

- Let $G Y=1$. Then $G^{\circ} X \cong 0$.

Proof.

$$
\begin{aligned}
G^{\circ} X & =\int_{Y} 1 \Rightarrow(X \times Y) \\
& \cong \int_{Y} X \times Y \\
& \cong X \times \int_{Y} Y \\
& \cong X \times 0 \\
& \cong 0
\end{aligned}
$$

- Let $G Y=A \times G^{\prime} Y$. Then $G^{\circ} X \cong A \Rightarrow G^{\circ} X$.

Proof.

$$
\begin{aligned}
G^{\circ} X & =\int_{Y} A \times G_{0} Y \Rightarrow(X \times Y) \\
& \cong \int_{Y} A \Rightarrow\left(G_{0} Y \Rightarrow(X \times Y)\right) \\
& \cong A \Rightarrow \int_{Y} G_{0} Y \Rightarrow(X \times Y) \\
& =A \Rightarrow G_{0}^{\circ} X
\end{aligned}
$$

- A little more generally, for $G Y=\sum a: A . G^{\prime} a Y$, one has $G^{\circ} X \cong \prod a: A .\left(G^{\prime} a\right)^{\circ} X$.
- Specializing to $A=0$ resp. $A=\mathbb{B}$, we learn: Let $G Y=0$. Then $G^{\circ} X \cong 1$. Let $G Y=G_{0} Y+G_{1} Y$. Then $G^{\circ} X \cong G_{0}^{\circ} X \times G_{1}^{\circ} X$.

Dual of exponents of the identity functor Let $G Y=A \Rightarrow Y$. Then $G^{\circ} X \cong A \times X$.
Proof.

$$
\begin{aligned}
G^{\circ} X & =\int_{Y}(A \Rightarrow Y) \Rightarrow(X \times Y) \\
& \cong X \times A \\
& \cong A \times X
\end{aligned}
$$

Example 4. Let $G Y=Y^{+}=\mu Z . Y \times(1+Z) \cong \sum n: \mathbb{N} .([0 . . n] \Rightarrow Y)$ (nonempty lists). We have $G^{\circ} X \cong \prod n: \mathbb{N} .([0 . . n] \times X)$.

Sometimes only a "lower bound" on the dual of a functor constructed from some given functors can be expressed in terms of their duals. This holds for the composition of two general functors, incl. for exponents of a general functor.

Dual of exponents of a general functor Let $G Y=A \Rightarrow G^{\prime} Y$. For a general $G^{\prime}$, we only have a canonical natural transformation with components $A \times G^{\circ} Y \rightarrow G^{\circ} Y$.

Proof.

$$
\begin{aligned}
G^{\circ} X & =\int_{Y}\left(A \Rightarrow G^{\prime} Y\right) \Rightarrow(X \times Y) \\
& \leftarrow \int_{Y} A \times\left(G^{\prime} Y \Rightarrow(X \times Y)\right) \\
& \cong A \times \int_{Y} G^{\prime} Y \Rightarrow(X \times Y) \\
& =A \times G^{\prime \circ} X
\end{aligned}
$$

Dual of composition of two general functors For general $G_{0}, G_{1}$, we only have the canonical natural transformation $\mathrm{m}^{G_{0}, G_{1}}: G_{0}^{\circ} \cdot G_{1}^{\circ} \rightarrow\left(G_{0} \cdot G_{1}\right)^{\circ}$.

Proof.

$$
\begin{aligned}
G^{\circ} X & =\int_{Y} G_{0}\left(G_{1} Y\right) \Rightarrow(X \times Y) \\
& \leftarrow \int_{Y} \int_{Z}\left(Z \Rightarrow G_{1} Y\right) \Rightarrow\left(G_{0} Z \Rightarrow(X \times Y)\right) \\
& \cong \int_{Z} G_{0} Z \Rightarrow \int_{Y}\left(Z \Rightarrow G_{1} Y\right) \Rightarrow(X \times Y) \\
& \leftarrow \int_{Z} G_{0} Z \Rightarrow \int_{Y}\left(G_{1} Y \Rightarrow(X \times Y)\right) \times Z \\
& \cong \int_{Z} G_{0} Z \Rightarrow\left(\left(\int_{Y} G_{1} Y \Rightarrow(X \times Y)\right) \times\left(\int_{Y} Z\right)\right) \\
& =\int_{Z} G_{0} Z \Rightarrow\left(G_{1}^{\circ} X \times Z\right) \\
& =G_{0}^{\circ}\left(G_{1}^{\circ} X\right)
\end{aligned}
$$

This hints that $(-)^{\circ}:[\mathcal{C}, \mathcal{C}]^{\text {op }} \rightarrow[\mathcal{C}, \mathcal{C}]$ is not monoidal, but only lax monoidal (see Section 3.4).
Example 5. Let $G_{0} Y=A \Rightarrow Y, G_{1} Y=B \times Y$, so $G Y=\left(G_{0} \cdot G_{1}\right) Y=A \Rightarrow(B \times Y) \cong$ $(A \Rightarrow B) \times(A \Rightarrow Y)$. The dual of $G$ is $G^{\circ} X \cong(A \Rightarrow B) \Rightarrow(A \times X)$ rather than $\left(G_{0}^{\circ} \cdot G_{1}^{\circ}\right) X \cong A \times(B \Rightarrow X)$ as we might perhaps expect. We saw the interaction law of $G$ with $G^{\circ}$ in Example 2. The canonical natural transformation $\mathrm{m}^{G_{0}, G_{1}}: G_{0}^{\circ} \cdot G_{1}^{\circ} \rightarrow G^{\circ}$ is $\mathrm{m}^{G_{0}, G_{1}}(a, f)=\lambda g .(a, f(g a))$.

## 3 Monad-comonad interaction

### 3.1 Monad-comonad interaction laws

In a monad-comonad interaction law, the allowed computations (the chosen notion of computation) must include "just returning" and be closed under sequential composition, so they are defined by a monad rather than a functor. To match this, the allowed machine behaviors (the notion of machine behavior) are defined by a comonad. The idea is that interaction of a "just returning" computation should terminate immediately (in the initial state of the given machine behavior) whereas interaction of a sequence of computations should amount to a sequence of interactions.

We define a monad-comonad interaction law on $\mathcal{C}$ to be given by a monad $T=(T, \eta, \mu)$ and a comonad $D=(D, \varepsilon, \delta)$ together with a family $\psi$ of maps

$$
\psi_{X, Y}: T X \times D Y \rightarrow X \times Y
$$

natural in $X$ and $Y$ (i.e., a functor-functor interaction law of $T, D$ where $T$ and $D$ carry a monad resp. comonad structure) such that also


Example 6. Take $T X=A \Rightarrow X, D Y=A \times Y$ and $\psi(f,(a, y))=(f a, y)$ for a fixed object $A$. The functors $T$ and $D$ are a monad (a reader monad) resp. a comonad and $\psi$ meets the conditions (1).

Example 7. Take $T X=B \times X, D Y=B \Rightarrow Y, \psi((b, x), g)=(x, g b)$ for a fixed monoid $B$. The functors $T, D$ are a monad (a writer monad) resp. a comonad and $\psi$ meets the requisite conditions.

Example 8. Take $T X=A \Rightarrow(B \times X), D Y=A \times(B \Rightarrow Y), \psi(f,(a, g))=$ let $(b, x) \leftarrow f a$ in $(x, g b)$ for a fixed monoid $B$ acting on a fixed object $A$. The functors $T, D$ are a monad (an update monad [6]) resp. a comonad and $\psi$ meets the requisite conditions.

Monad-comonad interaction laws are essentially the same as monoid objects in the monoidal category $\mathbf{I L}(\mathcal{C})$ of functor-functor interaction laws. To be precise, a monad-comonad interaction law $((T, \eta, \mu)$, $(D, \varepsilon, \delta), \psi)$ yields a monoid $((T, D, \psi),(\eta, \varepsilon),(\mu, \delta))$ and vice versa.

A monad-comonad interaction law map between $(T, D, \psi),\left(T^{\prime}, D^{\prime}, \psi^{\prime}\right)$ is a pair $\left(f: T \rightarrow T^{\prime}, g\right.$ : $\left.D^{\prime} \rightarrow D\right)$ of a monad map and a comonad map that, as a pair of natural transformations between the underlying functors, is a functor-functor interaction law morphism between the underlying functor-functor interaction laws.

Monad-comonad interaction law maps correspond to monoid morphisms in IL(C). Thus monadcomonad interaction laws form a category $\operatorname{MCIL}(\mathcal{C})$ isomorphic to the category $\operatorname{Mon}(\operatorname{IL}(\mathcal{C}))$.

### 3.2 A degeneracy result

Monads with an associative operation Here is a degeneracy theorem for monad-comonad interaction laws.

Theorem 3. If a monad $T$ has an associative binary operation, i.e., family of maps $c_{X}: X \times X \rightarrow T X$ natural in $X$ satisfying

then, for any comonad $D$ and interaction law $\psi_{X, Y}: T X \times D Y \rightarrow X \times Y$, we have


Proof. For any $Y$, by distributivity in an extensive category, $\mathbb{B} \times Y$ is a coproduct of $Y$ and $Y$ with coprojections $\langle\mathrm{tt} \circ$ !, id $\rangle$ and $\langle\mathrm{ff} \circ$ !, id $\rangle$.

By stability of coproducts under pullback in an extensive category, we can pull $\theta_{\mathbb{B}, Y} \circ\left\langle c_{\mathbb{B}} \circ\langle\mathrm{tt}, \mathrm{ff}\rangle \circ\right.$ !, id $\rangle$ : $D Y \rightarrow \mathbb{B} \times Y$ back along the coprojections of $\mathbb{B} \times Y$ and get that $D Y$ is a coproduct of two objects $P Y$ and $Q Y$ with coprojections $i_{Y}$ and $j_{Y}$ :


It is easily checked that we have


Also by stability of coproducts under pullback, we can pull $\delta_{Y}: D Y \rightarrow D D Y$ back along the coprojections of $D D Y$ and get that $D Y$ is a coproduct of two objects $P^{\prime} Y$ and $Q^{\prime} Y$ with coprojections $i_{Y}^{\prime}$ and
$j_{Y}^{\prime}$ :


Hence, for any $X$, by distributivity, also $X \times(X \times X) \times D Y$ is a coproduct of $X \times(X \times X) \times P^{\prime} Y$ and $X \times(X \times X) \times Q^{\prime} Y$ with coprojections id $\times i_{Y}^{\prime}$ and id $\times j_{Y}^{\prime}$.

Now, the two maps $\psi_{X, Y} \circ \psi_{T X, D Y} \circ\left(c_{T X} \circ \eta_{X} \times c_{X}\right) \times \delta_{Y}$ and $\psi_{X, Y} \circ \psi_{T X, D Y} \circ\left(c_{T X} \circ \eta_{X} \times\left(\eta_{X} \circ\right.\right.$ snd $\left.)\right) \times \delta_{Y}$ both satisfy both triangles of the unique copair of $\psi_{X, Y} \circ\left(\eta_{X} \circ f\right.$ ft $) \times\left(h_{D Y} \circ f_{Y}\right)$ and $\psi_{X, Y} \circ\left(c_{X} \circ\right.$ snd $) \times$ $\left(k_{D Y} \circ g_{Y}\right)$, i.e., they are the same map. Indeed, we have


And, using associativity, we also have


The desired result now follows by the following calculation:


Example 9. The monad $T X=X^{+}$of nonempty lists (the free semigroup delivering monad) comes with an associative operation $\mathrm{dblt}_{X}: X \times X \rightarrow T X$ defined by dblt $\left(x_{0}, x_{1}\right)=\left[x_{0}, x_{1}\right]$. The degeneracy theorem tells us that, while functor-functor interaction laws can accomplish this, no monad-comonad interaction law can extract $x_{1}$ from a list $\left[x_{0}, x_{1}, x_{2}\right]$ and more generally any middle element $x_{i}(0<i<n+1)$ from a list $\left[x_{0}, \ldots, x_{n+1}\right]$.

Just as functor-functor interaction laws can be generalized to a residual variant to counteract degeneracies, so can monad-comonad interaction laws (see Section 55).

### 3.3 On the structure of $\operatorname{MCIL}(\mathcal{C})$

We now explore the structure of the category $\operatorname{MCIL}(\mathcal{C})$. As this is the category of monoid objects of $\mathbf{I L}(\mathcal{C})$, the structure of $\operatorname{MCIL}(\mathcal{C})$ is in many respects similar to $\mathbf{I L}(\mathcal{C})$. But there are also important differences.
"Stretching" Given a monad-comonad interaction law $(T, D, \psi)$, a monad morphism $f: T^{\prime} \rightarrow T$ and a comonad morphism $g: D^{\prime} \rightarrow D$, we have a monad-comonad interaction law ( $T^{\prime}, D^{\prime}, \psi \circ f \times g$ ).

Final and initial monad-comonad interaction laws The final monad-comonad interaction law is $(1,0, \psi)$ where $\psi_{X, Y}: 1 \times 0 \rightarrow X \times Y$ is the evident map.

The initial monad-comonad interaction law is ( $\left.\mathrm{Id}, \mathrm{Id}, \mathrm{id}_{\mathbf{I d} \times \mathbf{I d}}\right)$.
Product of two monad-comonad interaction laws Given two monad-comonad interaction laws $\left(T_{0}, D_{0}, \psi_{0}\right)$ and $\left(T_{1}, D_{1}, \psi_{1}\right)$, their product is $\left(T_{0} \times T_{1}, D_{0}+D_{1}, \psi\right)$ where $\psi_{X, Y}:\left(T_{0} X \times T_{1} X\right) \times\left(D_{0} Y+D_{1} Y\right) \rightarrow X \times Y$ is defined as in Section 2. The product of the underlying functors of the two monads is the underlying functor of their product.

Coproduct of two monad-comonad interaction laws The coproduct of two monad-comonad interaction laws is given by the coproduct of the two monads, the product of the two comonads and a suitable natural transformation. The coproduct of two monads is complicated to construct. For two ideal monads, it can be expressed in terms of initial algebras of endofunctors on $\mathcal{C} \times \mathcal{C}$ (mutually inductive types) [14].

Interaction laws of a composite monad Given two monad-comonad interaction laws ( $T_{0}, D, \psi_{0}$ ) and $\left(T_{1}, D, \psi_{1}\right)$ and a monad-monad distributive law $\lambda$ of $T_{1}$ over $T_{0}$. Then $T_{0} \cdot T_{1}$ is a monad. If $\psi_{0}$ and $\psi_{1}$ are matching in the sense of commutation of

then we have a monad-comonad interaction law $\left(T_{0} \cdot T_{1}, D, \psi\right)$ where

$$
\psi_{X, Y}=T_{0} T_{1} X \times D Y \xrightarrow{\mathrm{id} \times \delta_{Y}} T_{0} T_{1} X \times D D Y \xrightarrow{\psi_{0} T_{1} X, D Y} T_{1} X \times D Y \xrightarrow{\psi_{1 X, Y}} X \times Y
$$

The condition above is precisely the condition for $\left(\lambda, \mathrm{id}_{D}\right)$ to be a map between the functor-functor interaction laws $\left(T_{1} \cdot T_{0}, D, \psi_{0} \circ \psi_{1} \cdot\left(T_{0} \times D\right) \circ\left(\mathrm{id}_{T_{0} \cdot T_{1}} \times \delta\right)\right)$ and $\left(T_{0} \cdot T_{1}, D, \psi_{1} \circ \psi_{0} \cdot\left(T_{1} \times D\right) \circ\left(\mathrm{id}_{T_{0} \cdot T_{1}} \times \delta\right)\right)$.

Interaction laws of a composite monad and a composite comonad Given two monad-comonad interaction laws $\left(T_{0}, D_{0}, \psi_{0}\right)$ and $\left(T_{1}, D_{1}, \psi_{1}\right)$, a monad-monad distributive law $\lambda$ of $T_{1}$ over $T_{0}$ and a comonadcomonad distributive law $\kappa$ of $D_{0}$ over $D_{1}$. Then $T_{0} \cdot T_{1}$ is a monad and $D_{0} \cdot D_{1}$ is a comonad. If $\psi_{0}$ and $\psi_{1}$ are matching in the sense of commutation of

then we have a monad-comonad interaction law $\left(T_{0} \cdot T_{1}, D_{0} \cdot D_{1}, \psi\right)$ where

$$
\psi_{X, Y}=T_{0} T_{1} X \times D_{0} D_{1} Y \xrightarrow{\psi_{0} T_{1} X, D_{1} Y} T_{1} X \times D Y \xrightarrow{\psi_{1 X, Y}} X \times Y
$$

The condition above is precisely the condition for $(\lambda, \kappa)$ to be a map between the functor-functor interaction laws $\left(T_{1} \cdot T_{0}, D_{1} \cdot D_{0}, \psi_{0} \circ \psi_{1} \cdot\left(T_{0} \times D_{0}\right)\right)$ and $\left(T_{0} \cdot T_{1}, D_{0} \cdot D_{1}, \psi_{1} \circ \psi_{0} \cdot\left(T_{1} \times D_{1}\right)\right)$.

An initial algebra-final coalgebra construction The initial algebra-final coalgebra construction from Section 2 gives a monad-comonad interaction law if we start with a parameterized monad $T$, a parameterized comonad $D 35$ and a family of maps $\psi_{X, Y, Z, W}: T(X, Z) \times D(Y, W) \rightarrow X \times Y+Z \times W$ natural in $X, Y, Z, W$ that agree in the sense of commutation of the diagrams



We get a monad-comonad interaction law $\left(T^{\prime}, D^{\prime}, \psi^{\prime}\right)$ where $T^{\prime} X=\mu Z . T(X, Z), D^{\prime} Y=\nu W \cdot D(Y, W)$ and $\psi^{\prime}$ is defined as in Section 2. The functors $T^{\prime}$ and $D^{\prime}$ carry monad resp. comonad structures 35] and the natural transformation $\psi$ agrees with those.

Free monad-comonad interaction law If $\mathcal{C}$ has relevant initial algebras and final coalgebras, then, given an interaction law $(F, G, \phi)$, the free monad-comonad interaction law is provided by the free monad $F^{*}$ and the cofree comonad $G^{\dagger}$ and a suitable natural transformation $\psi^{\prime}$.

The free monad is given by $F^{*} X=\mu Z . X+F Z$. Its monad structure is induced by the parameterized $\operatorname{monad} T(X, Z)=X+F Z$. Similarly, the cofree comonad is given by $G^{\dagger} Y=\nu W . Y \times G W$. Its comonad structure is induced by the parameterized comonad $D(Y, W)=Y \times G W$. In order to construct $\psi^{\prime}$ following the construction we described in the previous paragraph, we need to construct a family of maps $\psi_{X, Y, Z, W}:(X+F Z) \times(Y \times G W) \rightarrow X \times Y+Z \times W$ natural in $X, Y, Z, W$. This is defined as follows:

$$
\begin{aligned}
\psi_{X, Y, Z, W}=( & X+F Z) \times(Y \times G W) \xrightarrow{\text { ldist }} \\
& X \times(Y \times G W)+F Z \times(Y \times G W) \xrightarrow{\text { id } \times \mathrm{fst}+\mathrm{id} \times \mathrm{snd}} \\
& X \times Y+F Z \times G W \xrightarrow{\text { id }+\phi_{Z, W}} X \times Y+Z \times W
\end{aligned}
$$

Restricting to fixed $T$ or $D$ We denote the categories obtained from MCIL(C) by fixing the monad $T$ or the comonad $D$ by $\left.\operatorname{MCIL}(\mathcal{C})\right|_{T,-}$ and $\left.\operatorname{MCIL}(\mathcal{C})\right|_{-, D}$. The final object of $\left.\operatorname{MCIL}(\mathcal{C})\right|_{T,-}$ is $(T, 0, \psi)$ where $\psi_{X, Y}=T X \times 0 \stackrel{\text { snd }}{\rightarrow} 0 \stackrel{?}{\rightarrow} X \times Y$; note that 0 is the initial comonad. The initial object of MCIL $\left.(\mathcal{C})\right|_{-, D}$ is $(\mathrm{Id}, D, \psi)$ where $\psi_{X, Y}=X \times D Y \xrightarrow{\text { id } \times \varepsilon_{Y}} X \times Y$; this is because Id is the initial monad.

### 3.4 Monad-comonad interaction in terms of dual and Sweedler dual

Similarly to case of functor-functor interaction laws and maps between them, the dual allows us to obtain useful alternative characterizations of monad-comonad interaction laws and their maps. But a complication arises, see below $5^{5}$

First, let us notice that we have, canonically, a natural transformation e : Id $\rightarrow \mathrm{Id}^{\circ}$ and, for any $F, G$, a natural transformation $\mathrm{m}_{F, G}: F^{\circ} \cdot G^{\circ} \rightarrow(F \cdot G)^{\circ}$. These are informally defined by $\mathrm{e}_{X} x=\lambda_{Y} \cdot \lambda y \cdot(x, y)$ : $X \rightarrow \int_{Y} Y \Rightarrow(X \times Y)$ and $\left(\mathrm{m}_{F, G}\right)_{X} f=\lambda_{Y}$. $\lambda z$. let $(g, w) \leftarrow f_{G Y} z$ in $g_{Y} w: \int_{Y^{\prime}} F Y^{\prime} \Rightarrow\left(\int_{Y^{\prime \prime}} G Y^{\prime \prime} \Rightarrow\right.$ $\left.\left(X \times Y^{\prime \prime}\right)\right) \times Y^{\prime} \rightarrow \int_{Y} F(G Y) \Rightarrow(X \times Y)$. The natural transformation e is a natural isomorphism; its inverse $\mathrm{e}^{-1}: \mathrm{Id}^{\circ} \rightarrow$ Id is defined by $\mathrm{e}_{X}^{-1} f=$ let $\left(x,{ }_{-}\right) \leftarrow f 1 *$ in $x: \int_{Y} Y \Rightarrow(X \times Y) \rightarrow X$.

The data $(\mathrm{e}, \mathrm{m})$ satisfy the conditions to make $(-)^{\circ}:[\mathcal{C}, \mathcal{C}]^{\text {op }} \rightarrow[\mathcal{C}, \mathcal{C}]$ a lax monoidal functor wrt. the (Id, $\cdot$ ) composition monoidal structure of $[\mathcal{C}, \mathcal{C}]$.

Now, as a first alternative characterization, a monad-comonad interaction law of $T$ and $D$ is essentially the same as a natural transformation $\psi: T \rightarrow D^{\circ}$ satisfying


Now, since $(-)^{\circ}:[\mathcal{C}, \mathcal{C}]^{\text {op }} \rightarrow[\mathcal{C}, \mathcal{C}]$ is lax monoidal, it sends monoids in $[\mathcal{C}, \mathcal{C}]^{\text {op }}$ to monoids in $[\mathcal{C}, \mathcal{C}]$, i.e., comonads to monads. In particular, it sends the comonad $(D, \varepsilon, \delta)$ to the monad $D^{\circ}=\left(D^{\circ}, \varepsilon^{\circ} \circ \mathrm{e}, \delta^{\circ} \circ \mathrm{m}\right)$. The conditions above are precisely the conditions for $\psi$ to be a monad map from $T$ to $D^{\circ}$. Summing up, a monad-comonad interaction law of $T, D$ amounts to a monad map $\psi: T \rightarrow D^{\circ}$.

As a second alternative, a monad-comonad interaction law of $T, D$ is given by a natural transformation $\psi: D \rightarrow T^{\circ}$ satisfying


Now, unfortunately, $(-)^{\circ}$ is not oplax monoidal, so it does generally not send comonoids to comonoids, and $T^{\circ}$ is generally not a comonad. We could define a candidate counit for $T^{\circ}$ as $\mathrm{e}^{-1} \circ \eta^{\circ}: T^{\circ} \rightarrow \mathrm{Id}$, but there is generally no candidate for the comultiplication as we cannot invert $\mathrm{m}_{T, T}$. So we cannot generally say that a monad-comonad interaction law is a comonad map from $D$ to $T^{\circ}$; the functor $T^{\circ}$ is not a comonad.

But it may be that there exists what one could informally describe as the greatest comonad smaller (in an appropriate sense) than $T^{\circ}$. The formal object of interest here is what we call, following the use of this word in other contexts [3130|17, the Sweedler (or finite) dual of the monad $T$. It is really just the greatest among all comonads $D$ satisfying conditions (2).

We say that the Sweedler dual of the monad $T$ is the (unique up to isomorphism, if it exists) comonad $T^{\bullet}=\left(T^{\bullet}, \eta^{\bullet}, \mu^{\bullet}\right)$ together with a natural transformation $\iota: T^{\bullet} \rightarrow T^{\circ}$ such that

and such that, for any comonad $D=(D, \varepsilon, \delta)$ and a natural transformation $\psi$ satisfying conditions (22), there exists a unique comonad map $h: D \rightarrow T^{\bullet}$ satisfying $\psi=\iota \circ h$ as summarized in the following diagrams:


The left-hand diagrams of (3) and (2) are secondary in this definition. In the left-hand diagram of (3), $\eta^{\bullet}$ is determined by $\iota$ as $\eta^{\bullet}=\mathrm{e}^{-1} \circ \eta^{\circ} \circ \iota$. The left-hand diagram of 2 ) commutes trivially when $\psi=\iota \circ h$ for some comonad map $h$.

[^0]Now, if $T$ has the Sweedler dual, there is a bijection between monad-comonad interaction laws of $T$, $D$, i.e, natural transformations $\psi: D \rightarrow T^{\circ}$ satisfying $\sqrt{2}$, and comonad maps $h: D \rightarrow T^{\bullet}$. Indeed, any natural transformation $\psi$ satisfying (2) induces a unique comonad map $h$ such that $\iota \circ h=\psi$ by definition of $T^{\bullet}$. On the other hand, for a comonad map $h$, we get a natural transformation $\psi$ satisfying (2) simply as the composition $\iota \circ h$. These constructions are inverses.

To sum up, we have proved that the following categories are isomorphic:
(o) monad-comonad interaction laws;
(i) triples of a monad $T$, a comonad $D$ and a monad map from $T$ to $D^{\circ}$;
(ii) triples of a monad $T$, a comonad $D$ and a natural transformation from $D$ to $T^{\circ}$ subject to conditions (2);
(iii) triples of a monad $T$, a comonad $D$ and a comonad map from $D$ to $T^{\bullet}$.

We see that the initial object of $\left.\operatorname{MCIL}(\mathcal{C})\right|_{-, D}$ is (Id $, D, \ldots$ ) while the final object is $\left(D^{\circ}, D, \ldots\right)$. The initial object of $\left.\operatorname{MCIL}(\mathcal{C})\right|_{T,-}$ is $\left(T, T^{\bullet}, \ldots\right)$ while the final object is $(T, 0, \ldots)$.

Calculating the Sweedler dual is a complicated matter and we will come to it in Section 6. But here are two examples where the dual of the underlying functor of a monad is not a comonad and the underlying functor of the Sweedler dual differs from the dual.

Example 10. In Example 4, we saw that the dual of the functor $T X=X^{+}$(nonempty lists) was $T^{\circ} Y \cong \prod n: \mathbb{N} .[0 . . n] \times Y$. While the functor $T$ is a monad (the free semigroup delivering monad), its dual $T^{\circ}$ is not a comonad. The Sweedler dual is $T^{\bullet} Y=Y \times(Y+Y), \eta^{\bullet}\left(y,{ }_{-}\right)=y, \delta^{\bullet}\left(y\right.$, inl $\left.y^{\prime}\right)=$ $\left(\left(y, \operatorname{inl} y^{\prime}\right), \operatorname{inl}\left(y^{\prime}, \operatorname{inl} y^{\prime}\right)\right), \delta^{\bullet}\left(y, \operatorname{inr} y^{\prime}\right)=\left(\left(y, \operatorname{inr} y^{\prime}\right), \operatorname{inr}\left(y^{\prime}, \operatorname{inr} y^{\prime}\right)\right)$, with $\iota_{Y}: T^{\bullet} Y \rightarrow T^{\circ} Y$ defined by $\iota(y, \ldots) 0=(0, y), \iota\left(,, \operatorname{inl} y^{\prime}\right)(n+1)=\left(0, y^{\prime}\right), \iota\left(\right.$, inr $\left.y^{\prime}\right)(n+1)=\left(n+1, y^{\prime}\right)$. The monad-comonad interaction law $\psi_{X, Y}: T X \times T^{\bullet} Y \rightarrow X \times Y$ is defined by $\psi\left(\left[x_{0}\right],(y,-)\right)=\left(x_{0}, y\right)$, $\psi\left(\left[x_{0}, \ldots, x_{n+1}\right],\left(-, \operatorname{inl} y^{\prime}\right)\right)=\left(x_{0}, y^{\prime}\right), \psi\left(\left[x_{0}, \ldots, x_{n+1}\right],\left(-, \operatorname{inr} y^{\prime}\right)\right)=\left(x_{n+1}, y^{\prime}\right)$.

Example 11. We learned in Example 5 that the dual of the functor $T X=A \Rightarrow(B \times X)$ is $T^{\circ} Y=$ $(A \Rightarrow B) \Rightarrow(A \times Y)$. But the Sweedler dual of $T$ as a monad when $B$ is a monoid acting on $A$ is $T^{\bullet} Y=A \times(B \Rightarrow Y), \iota(a, f)=\lambda g \cdot(a, f(g a))$. In Example 8, we showed the monad-comonad interaction law of $T$ and $T^{\bullet}$.

## 4 Stateful running

Monad-comonad interaction laws are related to stateful runners as introduced by Uustalu 36. Next we present the basic facts about runners using the Sweedler dual and then explain the connection to monad-comonad interaction laws.

### 4.1 Runners

A runner is similar to a monad-comonad interaction law but the allowed machine behaviors are restricted to operate on a fixed state set and their dynamics is also fixed (in the sense that, for any prospective initial state, there is a behavior pre-determined). Only the initial state is not fixed. The state set is manifest but the notion of machine behavior and the pre-determined dynamics are coalesced with the interaction protocol into the natural transformation that is the runner. The runner is a polymorphic function sending any allowed computation and initial state into a return value and a final state.

Given a monad $T=(T, \eta, \mu)$ on $\mathcal{C}$, we call a (stateful) runner of $T$ an object $Y$ with a family $\theta$ of maps

$$
\theta_{X}: T X \times Y \rightarrow X \times Y
$$

natural in $X$, satisfying


Example 12. We revisit Example 5 about the update monad $T X=A \Rightarrow(B \times X)$ defined by an action $\downarrow: A \times B \rightarrow A$ of a monoid $B$ on an object $A$. An update lens [5] is an object $Y$ together with maps $l k p: Y \rightarrow A$, upd : Y×B $\rightarrow Y$ such that $l k p$ is a map between the $B$-sets ( $Y, u p d$ ) and $(A, \downarrow)$. Any update lens gives us a runner of $T$ via $\theta_{X}:(A \Rightarrow(B \times X)) \times Y \rightarrow X \times Y$ defined by $\theta(f, y)=$ let $(b, x) \leftarrow$ $f(l k p y)$ in $(x, \operatorname{upd}(y, b))$. In fact, runners of this monad are in a bijection with update lenses and those in turn are essentially the same as coalgebras for the comonad $D Y=A \times(B \Rightarrow Y)$.

A runner map between $(Y, \theta),\left(Y^{\prime}, \theta^{\prime}\right)$ is a map $f: Y \rightarrow Y^{\prime}$ satisfying $\left(\mathrm{id}_{X} \times f\right) \circ \theta_{X}=\theta_{X}^{\prime} \circ\left(\mathrm{id}_{T X} \times f\right)$. Runners and their maps form a category $\operatorname{Run}(T)$.

Like monad-comonad interaction laws and maps between them, runners and maps between them admit a number of alternative characterizations.

The first one is that runners of $T$ are essentially the same as objects $Y$ endowed with a monad map $\vartheta: T \rightarrow \mathrm{St}^{Y}$ where $\mathrm{St}^{Y}=\left(\mathrm{St}^{Y}, \eta^{Y}, \mu^{Y}\right)$ is the state monad for $Y$ whose underlying functor is defined by $\mathrm{St}^{Y} X=Y \Rightarrow(X \times Y)$. This is via the bijection of natural transformations

$$
\int_{X} \mathcal{C}(T X \times Y, X \times Y) \cong \int_{X} \mathcal{C}(T X, \underbrace{Y \Rightarrow(X \times Y)}_{\mathrm{st}^{Y} X})
$$

Under this bijection, the runner conditions amount to the monad map conditions


A map $f: Y \rightarrow Y^{\prime}$ is a runner map between $(Y, \vartheta),\left(Y^{\prime}, \vartheta^{\prime}\right)$ iff


Second, a runner of the monad $T$ is also essentially the same thing as a coalgebra $(Y, \gamma)$ of the functor $T^{\circ}$ satisfying the conditions


This is because of the bijection

$$
\int_{X} \mathcal{C}(T X \times Y, X \times Y) \cong \int_{X} \mathcal{C}(Y \times T X, Y \times X) \cong \mathcal{C}(Y, \underbrace{\int_{X} T X \Rightarrow(Y \times X)}_{T^{\circ} Y})
$$

A runner map between $(Y, \gamma),\left(Y^{\prime}, \gamma^{\prime}\right)$ is a coalgebra map, i.e., a map $f: Y \rightarrow Y^{\prime}$ such that


Recall that the functor $T^{\circ}$ is generally not a comonad as $\mathrm{m}_{T, T}$ is not invertible, so we cannot generally speak of functor coalgebras satisfying conditions (4) as comonad coalgebras.

Lastly, recall that the costate comonad for an object $Y$ is defined by $\operatorname{Cost}^{Y}=\left(\operatorname{Cost}^{Y}, \varepsilon^{Y}, \delta^{Y}\right)$ is defined by $\operatorname{Cost}^{Y} Z=(Y \Rightarrow Z) \times Y, \varepsilon^{Y}(f, y)=f y, \delta^{Y}(f, y)=\left(\lambda y^{\prime} .\left(f, y^{\prime}\right), y\right)$. This gives us a third characterization: a runner is essentially the same as an object $Y$ together with a natural transformation $\zeta$ between the underlying functor of the costate comonad $\operatorname{Cost}^{Y}$ and the functor $T^{\circ}$ satisfying



This is because of the bijection

$$
\mathcal{C}\left(Y, T^{\circ} Y\right) \cong \int_{Z} \mathcal{C}\left(Y \Rightarrow Z, Y \Rightarrow T^{\circ} Z\right) \cong \int_{Z} \mathcal{C}(\underbrace{(Y \Rightarrow Z) \times Y}_{\text {cost }^{Y} Z}, T^{\circ} Z)
$$

A runner map between $(Y, \zeta),\left(Y^{\prime}, \zeta^{\prime}\right)$ is a map $f: Y \rightarrow Y^{\prime}$ satisfying

If the Sweedler dual comonad $T^{\bullet}$ of the monad $T$ exists, then we can continue this reasoning. We see that a runner is the essentially the same as an object $Y$ with a comonad morphism between Cost ${ }^{Y}$ and $T^{\bullet}$ and that is further essentially the same as an object $Y$ with a comonad coalgebra of $T^{\bullet}$.

Summing up, we have established that the following categories are isomorphic:
(o) runners of $T$;
(i) objects $Y$ with a monad map from $T$ to $\mathrm{St}^{Y}$;
(ii) functor coalgebras of $T^{\circ}$ subject to conditions (4);
(iii) objects $Y$ with a natural transformation from Cost $^{Y}$ to $T^{\circ}$ subject to conditions (5);
(iv) objects $Y$ with a comonad map from $\operatorname{Cost}^{Y}$ to $T^{\bullet}$;
(v) comonad coalgebras of $T^{\bullet}$.

### 4.2 Runners vs. monad-comonad interaction laws

Monad-comonad interaction laws of $T, D$ are in a bijection with $D$-coalgebraic $T$-runner specs by which we mean carrier-preserving functors between $\operatorname{Coalg}(D)$ and $\operatorname{Run}(T)$, i.e., functors $\Psi: \operatorname{Coalg}(D) \rightarrow \boldsymbol{\operatorname { R u n }}(T)$ such that


Indeed, given a monad-comonad interaction law $\psi$, we can define a runner spec $\Psi$ by

$$
(\Psi(Y, \gamma))_{X}=\left(Y, T X \times Y \xrightarrow{i \mathrm{~d} \times \gamma} T X \times D Y \xrightarrow{\psi_{X, Y}} X \times Y\right)
$$

In the opposite direction, given a runner spec $\Psi$, we build a interaction law from the cofree coalgebras of $D$. For any $Y$, we have the cofree coalgebra $\left(D Y, \delta_{Y}\right)$ and define a monad-comonad interaction law $\phi$ by

$$
\phi_{X, Y}=T X \times D Y \xrightarrow{\Psi\left(D Y, \delta_{Y}\right)_{X}} X \times D Y \xrightarrow{\mathrm{id} \times \varepsilon_{Y}} X \times Y
$$

A pair of a monad map $f: T \rightarrow T^{\prime}$ and a comonad map $g: D^{\prime} \rightarrow D$ is an interaction law map between $(T, D, \psi)$ and $\left(T^{\prime}, D^{\prime}, \psi^{\prime}\right)$ iff the corresponding coalgebraic runner specs satisfy

(Notice that $\operatorname{Coalg}(-): \operatorname{Comnd}(\mathcal{C}) \rightarrow \mathbf{C A T}$ and $\operatorname{Run}(-):(\operatorname{Mnd}(\mathcal{C}))^{\text {op }} \rightarrow \mathbf{C A T}$.) So the categories of monad-comonad interaction laws and coalgebraic runner specs are isomorphic.

More modularly, but assuming that all Sweedler duals exist, the isomorphism of the categories of monad-comonad interaction laws and coalgebraic runner specs follows from the following sequence of isomorphisms of categories, using that $\operatorname{Run}(T) \cong \operatorname{Coalg}\left(T^{\bullet}\right)$ :
(o) monad-comonad interaction laws;
(i) triples of a monad $T$, a comonad $D$ and a comonad map between $D, T^{\bullet}$;
(ii) triples of a monad $T$, a comonad $D$ and a carrier-preserving functor between $\operatorname{Coalg}(D), \operatorname{Coalg}\left(T^{\bullet}\right)$;
(iii) coalgebraic runner specs.

[^1]
## 5 Residual interaction and running

We will now generalize interaction laws to allow that that not all of the effect of a computation is serviced by a machine behavior in an interaction.

### 5.1 Residual interaction

Given a monad $R=\left(R, \eta^{R}, \mu^{R}\right)$ on our base category $\mathcal{C}$. We can generalize functor-functor and monadcomonad interaction laws as follows.

An $R$-residual functor-functor interaction law is given by endofunctors $F, G$ on $\mathcal{C}$ together with a family of maps

$$
\phi_{X, Y}: F X \times G Y \rightarrow R(X \times Y)
$$

natural in $X, Y$.
An $R$-residual interaction law map between $(F, G, \phi),\left(F^{\prime}, G^{\prime}, \phi^{\prime}\right)$ is given by natural transformations $f: F \rightarrow F^{\prime}, g: G^{\prime} \rightarrow G$ such that

$R$-residual functor-functor interaction laws form a category $\operatorname{IL}(\mathcal{C}, R)$.
This category is monoidal. The tensorial unit is (Id, Id, $\eta^{R} \cdot(\mathrm{Id} \times \mathrm{Id})$ ). The tensor of $(F, G, \phi)$ and $(J, K, \psi)$ is $\left(F \cdot J, G \cdot K, \mu^{R} \circ R \cdot \psi \circ \phi \cdot(J \times K)\right)$.

An $R$-residual monad-comonad interaction law of a monad $T$ and a comonad $D$ is a family $\psi$ of maps

$$
\psi_{X, Y}: T X \times D Y \rightarrow R(X \times Y)
$$

natural in $X$ and $Y$, satisfying



Example 13. Let $R X=X+E$ (the exceptions monad). Take $T X=A \Rightarrow(X+E), D Y=A \times Y$; these are a monad and a comonad. The natural transformation $\psi(f,(a, y))=$ case $f a$ of (inl $x \mapsto($ inl $x, y) \mid$ inr $e \mapsto \operatorname{inr} e)$ satisfies the conditions of a $R$-residual monad-comonad interaction law.
$R$-residual monad-comonad interaction laws are the same as monoid objects in the monoidal category $\mathbf{I L}(\mathcal{C}, R)$.
$R$-residual monad-comonad interaction law maps are defined as expected and correspond to monoid morphisms.

The category $\operatorname{MCIL}(\mathcal{C}, R)$ of $R$-residual monad-comonad interaction laws is isomorphic to $\operatorname{Mon}(\operatorname{IL}(\mathcal{C}, R))$.

### 5.2 Relationship to interaction laws on Kleisli categories

It is tempting to guess that an $R$-residual functor-functor interaction law of $F, G$ would be the same thing as a functor-functor interaction law on the Kleisli category of $R$. But this is jumping to conclusions too hastily. For something like this to be feasible, we need, first of all, that $F, G$ lift to $\mathbf{K l}(R)$. A necessary and sufficient condition is the presence of distributive laws of $F$ and $G$ over $R$, i.e., natural transformations $\kappa: F \cdot R \rightarrow R \cdot F$ and $\lambda: G \cdot R \rightarrow R \cdot G$ agreeing with the monad structure of $R$. Then we define the
lifted versions of $F, G$ on objects by $\bar{F} X=F X, \bar{G} Y=G Y$; for maps $k: X \rightarrow R X^{\prime}, \ell: Y \rightarrow R Y^{\prime}$, we define $\bar{F} k=\kappa_{X^{\prime}} \circ F k: F X \rightarrow R F X^{\prime}$ and $\bar{G} \ell=\lambda_{Y^{\prime}} \circ G \ell: G Y \rightarrow R G Y^{\prime}$.

Moreover, we also need to lift $\times$ to $\mathbf{K l}(R)$ as a bifunctor and monoidal structure. For this, a necessary and sufficient condition is monoidality of $R$ as a monad, i.e., the presence of a family of maps $m_{X, Y}$ : $R X \times R Y \rightarrow R(X \times Y)$ natural in $X, Y$ agreeing with both the product monoidal structure of $\mathcal{C}$ and the monad structure of $R$. (This is the same as $R$ being commutative strong monad.) For objects, we then define $X \overline{\times} Y=X \times Y$, and for maps $k: X \rightarrow R X^{\prime}, \ell: Y \rightarrow R Y^{\prime}$, we define $k \overline{\times} \ell=m_{X^{\prime}, Y^{\prime}} \circ(k \times \ell)$ : $X \times Y \rightarrow R\left(X^{\prime} \times Y^{\prime}\right)$.

The naturality condition for $\phi_{X, Y}: F X \times G Y \rightarrow R(X \times Y)$ as an interaction law of $\bar{F}, \bar{G}$ is: for all $k, \ell$,


But the naturality condition for $\phi$ as an $R$-residual interaction law of $F, G$ is: for all $f, g$,

$$
\begin{gathered}
F X \times G Y \xrightarrow{\phi_{X, Y}} R(X \times Y) \\
F f \times G g \downarrow \begin{array}{|}
\downarrow \\
F X^{\prime} \times G Y^{\prime} \xrightarrow{\phi_{X^{\prime}, Y^{\prime}}} R\left(X^{\prime} \times Y^{\prime}\right)
\end{array} \\
\left.F X^{\prime} \times g\right)
\end{gathered}
$$

The first condition implies the second:


The second condition gives the first condition restricted to pure maps of $\mathbf{K l}(R)$ (maps in the image of the left adjoint $J$ the Kleisli adjunction of $R$ ), i.e., for maps $k$, $\ell$ of the form $k=J f=\eta_{X^{\prime}} \circ f$,

$$
\ell=J g=\eta_{Y^{\prime}} \circ g:
$$



We thus see that $R$-residual functor-functor interaction laws are more liberal than functor-functor interaction laws in $\mathbf{K l}(R)$ in that we do not need the distributive laws and monoidality of $R$ and that the naturality condition is weaker (only required for pure maps).

### 5.3 Residual stateful running

Similarly to interaction laws, the concept of runners can also be generalized.
Given a monad $R=\left(R, \eta^{R}, \mu^{R}\right)$ on $\mathcal{C}$. An $R$-residual runner of a monad $T=(T, \eta, \mu)$ on $\mathcal{C}$ is an an object $Y$ with a family $\theta$ of maps

$$
\theta_{X}: T X \times Y \rightarrow R(X \times Y)
$$

natural in $X$, satisfying


A map of $R$-residual runners of $T$ between $(Y, \theta),\left(Y^{\prime}, \theta^{\prime}\right)$ is a map $f: Y \rightarrow Y^{\prime}$ satisfying

$R$-residual runners of $T$ form a category $\operatorname{Run}(T, R)$.
$R$-residual runners of $T$ are essentially the same as objects $Y$ endowed with a monad map $\vartheta: T \rightarrow$ $\mathrm{St}^{R, Y}$ where $\mathrm{St}^{R, Y}=\left(\mathrm{St}^{R, Y}, \eta^{R, Y}, \mu^{R, Y}\right)$ is the $R$-transformed state monad for $Y$ whose underlying functor is defined by $\mathrm{St}^{R, Y} X=Y \Rightarrow R(X \times Y)$.

This is via the bijection of natural transformations

$$
\int_{X} \mathcal{C}(T X \times Y, R(X \times Y)) \cong \int_{X} \mathcal{C}(T X, \underbrace{Y \Rightarrow R(X \times Y)}_{\mathrm{St}^{R, Y} X})
$$

Under this bijection, the $R$-residual runner conditions amount to the monad map conditions


A map $f: Y \rightarrow Y^{\prime}$ is a map of $R$-residual runners of $T$ between $(Y, \vartheta),\left(Y^{\prime}, \vartheta^{\prime}\right)$ iff


So the categories of $R$-residual runners of $T$ and objects $Y$ equipped with a monad map from $T$ to $\mathrm{St}^{R, Y}$ are isomorphic.

## 6 Monoid-comonoid interaction

Exploiting that monads and monad-like objects like arrows or lax monoidal functors ("applicative functors") are monoids has turned out to be very fruitful in categorical semantics (see, e.g., [19|10|33]). We now explore this perspective by abstracting monad-comonad interaction laws into monoid-comonoid interaction laws. This leads us to further known concepts and methods from category theory.

### 6.1 Interaction laws and Chu spaces

The first step in generalizing interaction laws to monoids and comonoids is to account for interaction laws as maps in a category. Recall that the Day convolution 12 of functors $F, G: \mathcal{C} \rightarrow \mathcal{C}$ where $\mathcal{C}$ is a category with finite products is given by

$$
(F \star G) Z=\int^{X, Y} \mathcal{C}(X \times Y, Z) \bullet(F X \times G Y)
$$

provided that this coend exists. (We take the same stance toward the question of well-definedness of the Day convolution as we took toward the well-definedness of the dual in Section 2 , By reasoning about natural transformations, we see that interaction laws for a pair of functors $F$ and $G$ amount to maps $\phi: F \star G \rightarrow \mathrm{Id}_{\mathcal{C}}:$

$$
\int_{X, Y} \mathcal{C}(F X \times G Y, X \times Y) \cong \int_{X, Y, Z} \operatorname{Set}(\mathcal{C}(X \times Y, Z), \mathcal{C}(F X \times G Y, Z)) \cong \int_{Z} \mathcal{C}((F \star G) Z, Z)
$$

We see that a functor-functor interaction law is a triple $\left(F, G, \phi: F \star G \rightarrow \operatorname{Id}_{\mathcal{C}}\right)$, i.e., a Chu space [7] over the monoid object $\mathrm{Id}_{\mathcal{C}}$ wrt. the Day convolution monoidal structure on $[\mathcal{C}, \mathcal{C}]$. An interaction law map is a Chu space map under this view, so the category $\mathbf{I L}(\mathcal{C})$ is isomorphic to the category $\mathbf{C h u}\left([\mathcal{C}, \mathcal{C}], I_{\mathcal{C}}\right)$.

This is nice, but not fine-grained enough for developing an abstract foundation for our theory. The canonical monoidal structure on $\operatorname{Chu}(\mathcal{F}, R)$ (where $R$ is a monoid object in $\mathcal{F}$ ) is based on the monoidal structure of the base category $\mathcal{F}$, which in our case is the Day convolution, and uses pullbacks. But we are interested in a different monoidal structure on $\mathbf{I L}(\mathcal{C})$ that is based on composition and gives us monads and comonads as monoids resp. comonoids. We fix this mismatch by moving to one of the cousins of the Chu construction: glueing à la Hasegawa.

### 6.2 Interaction laws and Hasegawa's glueing

Hasegawa's glueing construction [16] works as follows. Given two monoidal categories $\mathcal{F}=\left(\mathcal{F}, I^{\mathcal{F}}, \otimes^{F}\right)$, $\mathcal{G}=\left(\mathcal{G}, I^{\mathcal{G}}, \otimes^{\mathcal{G}}\right)$ and a lax monoidal functor $\left((-)^{\circ}, \mathrm{e}, \mathrm{m}\right): \mathcal{G} \rightarrow \mathcal{F}$. The comma category $\mathcal{F} \downarrow(-)^{\circ}$ carries a monoidal structure given by:

$$
\begin{gathered}
I=\left(I^{\mathcal{F}}, I^{\mathcal{G}}, I^{\mathcal{F}} \xrightarrow{\mathrm{e}}\left(I^{\mathcal{G}}\right)^{\circ}\right) \\
(F, G, \phi) \otimes\left(F^{\prime}, G^{\prime}, \phi^{\prime}\right)=\left(F \otimes^{\mathcal{F}} F^{\prime}, G \otimes^{\mathcal{G}} G^{\prime}, F \otimes^{\mathcal{F}} F^{\prime} \xrightarrow{\phi \otimes^{\mathcal{F}} \phi^{\prime}} G^{\circ} \otimes^{\mathcal{F}} G^{\prime \prime} \xrightarrow{\mathrm{m}_{G, G^{\prime}}}\left(G \otimes^{\mathcal{G}} G^{\prime}\right)^{\circ}\right)
\end{gathered}
$$

Also, if $\mathcal{F}$ and $\mathcal{G}$ are closed and $\mathcal{G}$ has pullbacks, then $\mathcal{F} \downarrow(-)^{\circ}$ is closed.
An interesting case of this construction is when we start with a duoidal category $(\mathcal{F}, I, \otimes, J, \star)$ closed wrt. $\star$ [13]. This is a category with two monoidal structures, and among its data are a map $\chi: I \star I \rightarrow I$ and a family of maps $\xi_{F, F^{\prime}, G, G^{\prime}}:\left(F \otimes F^{\prime}\right) \star\left(G \otimes G^{\prime}\right) \rightarrow(F \star G) \otimes\left(F^{\prime} \star G^{\prime}\right)$ natural in $F, F^{\prime}, G, G^{\prime}$. Moreover, given a monoid $\left(R, \eta^{R}, \mu^{R}\right)$ in $(\mathcal{F}, I, \otimes)$, we define $(-)^{\circ}: \mathcal{F}^{\mathrm{op}} \rightarrow \mathcal{F}$ by $G^{\circ}=G \rightarrow R$. This
functor $(-)^{\circ}$ is lax monoidal wrt. the $(I, \otimes)$ monoidal structure, since as witnesses e, m of lax monoidality we have the curryings of

$$
\begin{gathered}
\mathrm{e}^{\prime}=I \star I \xrightarrow{\chi} I \xrightarrow{\eta^{R}} R \\
\mathrm{~m}_{G, G^{\prime}}^{\prime}=\left(G^{\circ} \otimes G^{\prime \circ}\right) \star\left(G \otimes G^{\prime}\right) \xrightarrow{\xi}\left(G^{\circ} \star G\right) \otimes\left(G^{\prime \circ} \star G^{\prime}\right) \xrightarrow{\mathrm{ev} \otimes \mathrm{ev}} R \otimes R \xrightarrow{\mu^{R}} R
\end{gathered}
$$

Since $\mathcal{F}, \mathcal{F}^{\mathrm{op}},(-)^{\circ}$ fulfill the assumptions of the glueing construction, $\mathcal{F} \downarrow(-)^{\circ}$ is monoidal. A monoid in this category is what we will recognize as a $R$-residual monoid-comonoid interaction law in the duoidal category $\mathcal{F}$.

To recover the usual functor-functor and monad-comonad interaction laws, we take $(\mathcal{F}, I, \otimes, J, \star,-\star)$ to be $[\mathcal{C}, \mathcal{C}]$ with its composition monoidal and Day convolution monoidal closed structures, and define $G^{\circ}=G \rightarrow$ Id. An object of $\mathcal{F} \downarrow(-)^{\circ}$ is a functor-functor interaction law while a monad-comonad interaction law is a monoid object of this category. We ignore the issue that $\star$ and $\rightarrow \star$ need not be welldefined everywhere on $[\mathcal{C}, \mathcal{C}]$. As we remarked before, this can be solved by restricting to a full subcategory of $[\mathcal{C}, \mathcal{C}]$ given by some class of functors that is closed under $\star$ and $\rightarrow \star$ (such as finitary functors, cf. [13]).

The notions of dual and Sweedler dual emerge as follows in this setting. When the $\star$ monoidal structure is symmetric, we also have

$$
\mathcal{F}^{\text {op }} \frac{(-)^{\circ}}{\frac{T}{(-)^{\text {oop }}}} \mathcal{F}
$$

since, for any $F, G \in|[\mathcal{C}, \mathcal{C}]|$,

$$
\mathcal{F}\left(F, G^{\circ}\right) \cong \mathcal{F}(F \star G, \mathrm{ld}) \cong \mathcal{F}(G \star F, \mathrm{Id}) \cong \mathcal{F}\left(G, F^{\circ}\right) \cong \mathcal{F}^{\mathrm{op}}\left(F^{\circ}, G\right)
$$

Because of this adjunction, we call $(-)^{\circ}$ the dual.
Since the functor $(-)^{\circ}$ is lax monoidal, it lifts to a functor between the respective categories of monoids (bear in mind that $\left.\operatorname{Mon}\left(\mathcal{F}^{\mathrm{op}}\right)=(\operatorname{Comon}(\mathcal{F}))^{\mathrm{op}}\right)$ :


However, its left adjoint $(-)^{\text {oop }}$ is only oplax monoidal, but not lax monoidal, so we cannot get a similar diagram for $(-)^{\circ \mathrm{op}}$. We want to find a substitute for this lifting, in particular, we want a left adjoint for the lifted $(-)^{\circ}$ :

$$
(\operatorname{Comon}(\mathcal{F}))^{\mathrm{op}} \underset{(-)^{\cdot{ }^{\text {op }}}}{\frac{(-)^{\circ}}{\leftrightarrows}} \operatorname{Mon}(\mathcal{F})
$$

We obtain not a natural isomorphism between two functors $(\operatorname{Comon}(\mathcal{F}))^{\mathrm{op}} \rightarrow \mathcal{F}$ as in diagram (6), but instead only a natural transformation $\iota:(-)^{\circ \mathrm{op}} \cdot U \rightarrow U \cdot(-)^{\bullet \text { op }}$ between two functors Mon $(\mathcal{F}) \rightarrow \mathcal{F}^{\mathrm{op}}$.


We call the functor $(-)^{\bullet}:(\operatorname{Mon}(F))^{\mathrm{op}} \rightarrow \mathbf{C o m o n}(F)$ the Sweedler dual.

### 6.3 Sweedler dual for some constructions of monoids

As we have seen in the setting of monad-comonad interaction laws, it is not always easy to find the Sweedler dual. In the remainder of this section, we focus on the cases of free monoids and free monoids quotiented by "equations" for one method to compute them.

Let $F^{*}$ be the free monoid on $F$. In this case, if the cofree comonoid on $F^{\circ}$ exists, then it is the Sweedler dual of $F^{*}$, i.e., we can show that $\left(F^{*}\right)^{\bullet}=\left(F^{\circ}\right)^{\dagger}$. This is seen from the following calculation:

$$
\begin{aligned}
& (\operatorname{Comon}(\mathcal{F}))^{\mathrm{op}}\left(\left(F^{\circ}\right)^{\dagger}, D\right) \cong \operatorname{Comon}(\mathcal{F})\left(D,\left(F^{\circ}\right)^{\dagger}\right) \cong \mathcal{F}\left(U D, F^{\circ}\right) \\
& \quad \cong \mathcal{F}^{\mathrm{op}}\left(F^{\circ}, U D\right) \cong \mathcal{F}\left(F,(U D)^{\circ}\right) \cong \mathcal{F}\left(F, U D^{\circ}\right) \cong \operatorname{Mon}(\mathcal{F})\left(F^{*}, D^{\circ}\right)
\end{aligned}
$$

This observation facilitates calculation of Sweedler duals of free monads (i.e., theories without equations). A natural question is to ask what happens in the presence of equations. Suppose that we have a monoid $T$ given as a coequalizer

$$
E^{*} \xlongequal[g^{L}]{f^{L}} F^{*} \longrightarrow T
$$

in $\operatorname{Mon}(\mathcal{F})$ where $(-)^{L}$ is the left transpose of the free/forgetful adjunction between $\mathcal{F}$ and $\operatorname{Mon}(\mathcal{F})$. The maps $f, g: E \rightarrow U F^{*}$ of $\mathcal{F}$ represent a system of equations in a set of variables $E$, and we can think of $T$ as being the monoid obtained by calculating the free monoid and then quotienting by the equations. We can try to obtain the Sweedler dual of $T$ by constructing a "dual" diagram as follows. We can instantiate $\iota$ at $F^{*}$ and obtain a map $\iota_{F^{*}}:\left(U F^{*}\right)^{\circ} \rightarrow U\left(\left(F^{*}\right)^{\bullet}\right)$ in $\mathcal{F}^{\text {op }}$, i.e., a map $\iota_{F^{*}}: U\left(\left(F^{*}\right)^{\bullet}\right) \rightarrow\left(U F^{*}\right)^{\circ}$ in $\mathcal{F}$. By composing with $f^{\circ}$ and $g^{\circ}$, we get:

$$
U\left(\left(F^{\circ}\right)^{\dagger}\right)=U\left(\left(F^{*}\right)^{\bullet}\right) \xrightarrow[\iota^{\circ}]{\iota_{F^{*}}}\left(U F^{*}\right)^{\circ} \xlongequal[g^{\circ}]{\longrightarrow} E^{\circ}
$$

The Sweedler dual $T^{\bullet}$ of $T$ is now obtained as an equalizer in $\operatorname{Comon}(\mathcal{F})$ by

$$
T^{\bullet} \longrightarrow\left(F^{\circ}\right)^{\dagger} \xrightarrow[\left(g^{\circ} \circ \iota_{F^{*}}\right)^{R}]{\stackrel{\left(f^{\circ} \circ \iota_{F^{*}}\right)^{R}}{\Longrightarrow}}\left(E^{\circ}\right)^{\dagger}
$$

where $(-)^{R}$ is the right transpose of the forgetful/cofree adjunction between $\operatorname{Comon}(\mathcal{F})$ and $\mathcal{F}$.
Example 14. Revisiting Example 10, the nonempty list monad $T X=X^{+}$arises as the quotient of the free monad $T_{0} X=\mu Z . X+Z \times Z$ by the associativity equation for its operation $c_{X}: X \times X \rightarrow T_{0} X$, i.e., the equation


The monad $T_{0}$ is the free monad on the functor $F X=X \times X$. The dual of $F$ is $G Y=Y+Y$. The Sweedler dual of $T$ is the subcomonad of the cofree comonad $T_{0}^{\bullet} Y=\nu W . Y \times(W+W)$ by the coassociativity coequation for its cooperation $c_{Y}^{\prime}: T_{0}^{\bullet} Y \rightarrow Y+Y$, i.e., the coequation


With some calculation, we can find that $\left.T^{\bullet} Y \cong Y \times(Y+Y)\right]^{7}$ The comonad map $i: T^{\bullet} \rightarrow T_{0}^{\bullet}$ is defined by $i\left(y, \operatorname{inl} y^{\prime}\right)=\left(y, \operatorname{inl}\left(i\left(y^{\prime}, \operatorname{inl} y^{\prime}\right)\right)\right), i\left(y, \operatorname{inr} y^{\prime}\right)=\left(y, \operatorname{inr}\left(i\left(y^{\prime}, \operatorname{inr} y^{\prime}\right)\right)\right)$,

Compared to $T_{0}^{\bullet}$, the comonad $T^{\bullet}$ is relatively degenerate because coassociativity entails corectangularity (while associativity does not entail rectangularity), as the following theorem shows.
Theorem 4. Given a comonad $(D, \varepsilon, \delta)$ on $\mathcal{C}$ with a cooperation $c_{Y}: D Y \rightarrow Y+Y$. We show that the coequation of coassociativity


[^2]implies left and right corectangularity, i.e. the two coequations


Proof. We can pull $c_{D Y} \circ \delta_{Y}$ back along the coproduct coprojections (the existence of these pullbacks is part of extensivity):


By stability of coproducts under pullback (which is also part of extensivity), ( $\left.D Y, i_{Y}, j_{Y}\right)$ is a coproduct $P Y$ and $Q Y$.

For right corectangularity, we notice that the two maps $\left(\varepsilon_{Y}+c_{Y}\right) \circ c_{D Y} \circ \delta_{Y}$ and $(Y+\mathrm{inr}) \circ\left(\varepsilon_{Y}+\right.$ $\left.\varepsilon_{Y}\right) \circ c_{D Y} \circ \delta_{Y}$ both satisfy both triangles of the unique copair of inl $\circ \varepsilon_{Y} \circ f_{Y}$ and inr$\circ c_{Y} \circ g_{Y}$, so they must be the same map. Indeed, we have both

and, using coassociativity, also


The result now follows from noticing that $\left(\varepsilon_{Y}+\varepsilon_{Y}\right) \circ c_{D Y} \circ \delta_{Y}=c_{Y}$ :


Left corectangularity is proved analogously.

Example 15. Going back to Example 11, the update monad $T X=A \Rightarrow(B \times X)$ with $B=(B, \varnothing, \oplus)$ a monoid and $(A, \downarrow)$ a $B$-set arises as the quotient of the monad $T_{0} X=\mu Z \cdot X+(A \Rightarrow Z)+(B \times Z)$ by the following three equations for its operations $c_{X}:(A \Rightarrow X) \rightarrow T_{0} X$ and $d_{X}: B \times X \rightarrow T_{0} X$ :


The monad $T_{0} X$ is the free monad on the functor $F X=(A \Rightarrow X)+B \times X$. The dual of $F$ is $G Y=$ $(A \times Y) \times(B \Rightarrow Y)$. The Sweedler dual of the monad $T$ is the subcomonad of the cofree comonad $T_{0}^{\bullet} Y=\nu W . Y \times(A \times W) \times(B \Rightarrow W)$ on the functor $G$ resulting from imposing the following coequations on its cooperations $c_{Y}^{\prime}: T_{0}^{\bullet} Y \rightarrow A \times Y$ and $d_{Y}^{\prime}: T_{0}^{\bullet} Y \rightarrow B \Rightarrow Y$ :




Calculating, we can find that $T^{\bullet} Y \cong A \times(B \Rightarrow Y){ }^{9}$ The comonad map $i: T^{\bullet} \rightarrow T_{0}^{\bullet}$ is defined by $i(a, f)=\left(f \varnothing,(a, i(a, f)), \lambda b . i\left(a \downarrow b, \lambda b^{\prime} . f(b \oplus b)\right)\right)$.

## 7 Related work

Works closest related to this paper on monad-comonad interaction laws are Power and Shkaravska's work on arrays (lenses) as comodels [32, Power and Plotkin's study of tensors of models and comodels [28, Abou-Saleh and Pattinson's work on comodels for operational semantics [1], Møgelberg and Staton's work on linear usage of state [22] and Uustalu's work on runners [36] - the starting point for this work. Pattinson and Schröder [24] studied equational reasoning about comodels and noted the degeneracy from nullary and binary cocommutative cooperations; see also Bauer's tutorial [8]. Runners share some features with Plotkin and Pretnar's algebraic effect handlers [29], we describe them in the end of this section. In their new work 4, Ahman and Bauer proposed a language design for (residual) runners.

Hancock and Hyvernat's work on interaction structures [15] centers on the canonical interaction law of the free monad on $F^{\circ}$ and the cofree comonad on $F$ where $F$ is a container functor. (Intuitionistic) linearlogic based two-party session typing [34] is very much about canonical interaction between syntactically dual functors, as we discuss in the end of this section. The same idea is central in game-theoretic semantics of (intuitionistic) linear logic (formulae-as-games, proofs-as-strategies) [2].

[^3]Runners vs. handlers There are some similarities between handlers (now often called deep handlers) to runners, but also significant differences.

Given a monad $T=(T, \eta, \mu)$ on $\mathcal{C}$. We are interested in handling or running computations specified by $T$.

A handler [29] for an object (value set for input computations) $X$ is mathematically an algebra of the monad $T$ on $X$, i.e., an object $Z$ (return value set) with a map $\alpha: T Z \rightarrow Z$ satisfying the conditions of a monad algebra, that also comes with a map $f: X \rightarrow Z$.

A handler induces a unique map $h: T X \rightarrow Z$ satisfying

as $\left(\left(T X, \mu_{X}\right), \eta_{X}\right)$ is the free algebra of $T$ on $X$.
A runner, as we know, can be taken to be an object $Y$ (state set) with a monad map from $T$ to the state monad $\mathrm{St}^{Y}=\left(\mathrm{St}^{Y}, \eta^{Y}, \mu^{Y}\right)$, i.e., a natural transformation $\vartheta: T \rightarrow \mathrm{St}^{Y}$ such that, for any $X$, we have


We know that that such natural transformations $\vartheta$ are in a bijection with coalgebras of the comonad $T^{\bullet}$ with carrier $Y$, i.e., maps $\gamma: Y \rightarrow T^{\bullet} Y$ satisfying the conditions of a comonad coalgebra.

We can see that a handler induces a map $h$ with domain $T X$ where $X$ an arbitrary fixed object; the codomain of $h$ can be anything- $Z$ is an arbitrary fixed object. A runner, at the same time, is a family of maps $\vartheta_{X}$ with domain $T X$ where $X$ can be varied to be any object. The codomain of $\vartheta_{X}$ is of a prescribed form - it has to be $\mathrm{St}^{Y} X$ where $Y$ is an arbitrary fixed object. The map $h$ is induced by an algebra of the monad $T$ while the family of maps $\vartheta_{X}$ is induced by (and also induces) a coalgebra of the comonad $T^{\bullet}$.

We can make this comparison fairer by acknowledging that algebras $(Z, \alpha)$ of the monad $T$ are in a bijection with monad maps from $T$ to the continuations monad (Cont ${ }^{Z}, \eta^{Z}, \mu^{Z}$ ) for the answer set $Z$, defined by $\operatorname{Cont}^{Z} X=(X \Rightarrow Z) \Rightarrow Z$. Then for any $X$ and $f: X \rightarrow Z$, the map $h$ from diagram (7) factorizes as

where $\xi: T \rightarrow$ Cont $^{Z}$ is the monad map corresponding to the algebra structure $\alpha: T Z \rightarrow Z$.
Now both the handler-induced function $\xi$ and the runner $\vartheta$ are functions with domain $T X$ polymorphic in $X$. Still the handler-induced function $\xi$ is specified by an algebra of the monad $T$ while the runner $\vartheta$ is specified by a coalgebra of the comonad $T^{\bullet}$.

Conceptually, handlers and runners/interaction laws are really different in that, in the case of handlers, effects are treated inside a computation while runners/interaction laws use an outside machine to do this.

Session types In session type systems, one usually works with an inductively defined set of types along the lines of

| $G:=Y$ | return |
| :---: | :---: |
| $G_{0}+G_{1}$ | internal choice |
| $G_{0} \times G_{1}$ | external choice |
| $A \times G_{0}$ | output |
| $A \Rightarrow G_{0}$ | input |

where $Y$ is a type variable and $A$ is a base type. (For simplicity, we ignore inductive and coinductive types here.) Internal choice and external choice are really just special cases of output resp. input for $A=\mathbb{B}$.

The dual of a type is defined recursively by

$$
\begin{aligned}
Y^{\circ} & =Y \\
\left(G_{0}+G_{1}\right)^{\circ} & =G_{0}^{\circ} \times G_{1}^{\circ} \\
\left(G_{0} \times G_{1}\right)^{\circ} & =G_{0}^{\circ}+G_{1}^{\circ} \circ \\
\left(A \times G_{0}\right)^{\circ} & =A \Rightarrow G_{0}^{\circ} \\
\left(A \Rightarrow G_{0}\right)^{\circ} & =A \times G_{0}^{\circ}
\end{aligned}
$$

This syntactically defined dual agrees with our semantic concept of the dual of a functor, except for the last clause where a discrepancy arises. We work in an arbitrary Cartesian closed category. In session typing a linear setting is intended.

## 8 Conclusion and future work

We hope to have demonstrated that monad-comonad interaction laws are a natural concept for describing interaction of effectful computations with machines providing the effects. They are well-motivated not only as a computational model, but also mathematically, admitting an elegant theory based on concepts and methods that have previously proved useful in other mathematical contexts, such as the Sweedler dual.

There are many questions that we have not yet answered. What are some general ways to compute the Sweedler dual? Power's work [32] suggests a sophisticated iterative construction based on improving approximations. What is a good general syntax for cooperations and coequations? What can be said about the "dual" and the Sweedler "dual" in the presence of a residual monad and how to compute them in this situation? How to compute the Sweedler dual in some intuitionistic linear setting adequate for session typing?

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[^0]:    ${ }^{5}$ We discuss these isomorphisms of categories only on the level of objects here.

[^1]:    ${ }^{6}$ As $T^{\circ}$ is necessarily strong, we can apply an internal version of the Yoneda lemma.

[^2]:    ${ }^{7}$ This calculation was carried out in detail in 36.
    ${ }^{8}$ In band theory, left and right rectangularity are the equations $(x * y) * z=x * z$ and $x *(y * z)=x * z$.

[^3]:    ${ }^{9}$ Also this calculation appeared in 36.

