Partial Univalence in *n*-truncated Type Theory

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Abstract

It is well known that univalence is incompatible with uniqueness of identity proofs (UIP), the axiom that all types are h-sets. This is due to finite h-sets having non-trivial automorphisms as soon as they are not h-propositions.

A natural question is then whether univalence restricted to h-propositions is compatible with UIP. We answer this affirmatively by constructing a model where types are elements of a closed universe defined as a higher inductive type in homotopy type theory. This universe has a path constructor for simultaneous "partial" univalent completion, i.e., restricted to h-propositions.

More generally, we show that univalence restricted to (n-1)-types is consistent with the assumption that all types are n-truncated. Moreover we parametrize our construction by a suitably well-behaved container, to abstract from a concrete choice of type formers for the universe.

1 Introduction, Motivation, and Overview

Martin-Löf type theory [28] (MLTT) is a formal system useful both for dependently typed programming and as a foundations for the development of mathematics. It is the basis of proof assistants like Agda, Coq, Idris, Lean.

Homotopy type theory (HoTT) is a variation born out of the observation that equality proofs in MLTT behave like paths in homotopy theory [7]. A major focus is then to characterize the exact nature of equality for each type, filling some gaps left underspecified by MLTT by taking inpiration from the connection to spaces up to homotopy.

Central is Voevodsky's univalence axiom, stating that equalities of types corresponds to equivalence of types. From univalence other extensionality principles follow, like function and propositional extensionality: equality of functions corresponds to pointwise equality, and equality of propositions corresponds to logical equivalence.

Another important contribution is the introduction of higher inductive types (HITs), which generalize inductive types by not only allowing elements of the type but also equalities between them to be inductively generated. A general example is taking the quotient of a type by a relation, other examples are finite and countable powerset types[16, 35], ordinal notations [29], syntax of type theory up to judgemental equality [2], other forms of colimits, and types of spaces for synthetic homotopy theory [34]. Andrea Vezzosi Department of Computer Science IT University of Copenhagen Denmark avez@itu.dk

HoTT also brought attention to a classification of types based on the complexity of their equality type. We say that a type is (-2)-truncated or *contractible* if it is equivalent to the unit type, we say a type A is (n + 1)-truncated when for any x, y : A, the equality type $x =_A y$ is *n*-truncated. In particular (-1)-truncated types are referred to as *h*-propositions, and are those for which any two elements are equal, while 0-truncated types, whose equality types are h-propositions, are called *h*-sets.

The h-sets are the notion of set of homotopy type theory, and where most constructions will belong when using HoTT as a foundation for set-based mathematics or to reason about programs. Restricting oneself to types whose equality type is an h-proposition also avoids having to stipulate coherence conditions between different ways of proving the same equality. Such coherence conditions might be arbitrarily complex and not necessarily expressible within HoTT itself [22].

It would be tempting then, at least for these applications, to assume that every type is an h-set, i.e., the uniqueness of identity proofs (UIP). In the case of HITs, e.g. for set quotients, an explicit equality constructor can be included to impose the desired truncation level. However we are forced to step outside the h-sets when considering them collectively as a type, which we call the universe of h-sets, $\mathcal{U}^{\leq 0}$. In fact, by univalence, equalities in $\mathcal{U}^{\leq 0}$ correspond to isomorphisms between the equated h-sets, of which in general there are more than one. This is often unfortunate because of, e.g, the need to define sets by induction on a set quotient, or the lack of a convenient type that could take the role of a Grothendieck universe when formalizing categorical semantics in sets or presheaves.

The counterexample of $\mathcal{U}^{\leq 0}$ however does not apply to univalence restricted to h-propositions, i.e. proposition extensionality, since any two proofs of logical equivalence between two propositions can be proven equal. Moreover results about set-truncated HITs often rely on propositional extensionality when defining a map into h-propositions by induction. One example is effectiveness of quotients, i.e., that equalities $[a] =_{A/R} [b]$ between two representative of an equivalence class correspond to proofs of relatedness R(a, b). In this paper we show, for the first time, that UIP is consistent with univalence for h-propositions, and more generally that, for $n \ge 0$, the assumption that every type is *n*-truncated is consistent with univalence restricted to (n - 1)-truncated types (Corollary 4.8). We refer to this as *partial univalence*. We note that the result cannot be improved to include univalence for *n*-truncated types, as that would imply univalence for all types, and then we could prove that the *n*-th universe is not an *n*-type by the main result of [23].

We stress that existing truncated models such as the set model or groupoid model [19] do not model partial univalence. Although the set model contains a univalent universe of propositions, it is not the case that the set of small sets is univalent when restricted to propositions. Similarly, the groupoid model contains a univalent universe of sets, but the groupoid of small groupoids is not univalent when restricted to h-sets, i.e. groupoids with propositional sets of morphisms.

The main challenge will thus be how to interpret the universes of such a theory. For a fixed collection of type formers, we show how to overcome this in Section 2, where we construct a partially univalent universe through a indexed higher inductive type. In Section 3 we generalize the construction by a signature of type formers given as an indexed container [1]. We then prove the consistency of a partially univalent *n*-truncated type theory in Section 4. The proof uses a model of HoTT capable of interpreting indexed higher inductive types to derive a model for our theory.

The viability of MLTT as a programming language relies on the canonicity property: every closed term is equal to one in canonical form. While univalence as an axiom interferes with canonicity, cubical type theory [11] has remedied this by representing equality proofs as paths from an abstract interval type, and made univalence no longer an axiom. We formulate a partially univalent 0-truncated cubical type theory in Section 5. There we also prove that the theory satisfies homotopy canonicity, the property that every closed term is path equal to one in canonical form. We believe that this result establishes an important first step towards a computational interpretation of the theory. In Section 6 we discuss related works and conclude.

1.1 Formalization

We have formalized the main construction, the subject of Sections 2 and 3, in Cubical Agda. Separately, we have formalized Appendix A in Agda with univalence as a postulate. The formalizations are available as supplementary material to this article.

2 A 0-truncated Partially Univalent Universe of 0-types

In this section we set up some preliminary definitions and notations. We then provide a simpler case of our main technical result Theorem 3.13, to exemplify the reasoning necessary.

When reasoning internally in type theory, we write \equiv for judgmental equality and = for internal equality using the identity type. Given a type *A*, we also write the latter as A(-, -). Given $p : a_0 =_A a_1$ and a family *B* over *A*, the dependent equality $b_0 =_{B(p)} b_1$ of b_0 and b_1 over *p* is shorthand for the identity type $p_*(b_0) =_{B(a_1)} b_1$, where $p_*(b_0)$ is the transport of $b_0 : B(a_0)$ to $B(a_1)$ along *p*. Given a type *A* and $n \ge -2$, we write type_n(*A*) for the type that *A* is *n*-truncated. For n = -2, -1, 0, we have the usual special cases isContr(*A*), isProp(*A*), isSet(*A*) of *A* being contractible, propositional, and a set, respectively. All of these types are propositions.

We recall the notion of a univalent family.

Definition 2.1 (Univalence). A family *Y* over a type *X* is *univalent* if the canonical map $X(x_0, x_1) \rightarrow Y(x_0) \simeq Y(x_1)$ is an equivalence for all $x_0, x_1 : X$.

We relativize this notion with respect to a property P on types. This is supposed to be an extensional property, in the sense that it should depend only on Y(x), not on the "code" x : X. To make this precise, we let Y be a valued in a universe \mathcal{U} and express P as a propositional family over \mathcal{U} .

We use the word universe in a rather weak sense: until we add closure under some type formers, it can refer to an arbitrary type family. Notationwise, universes are distinguished in that we leave the decoding function from elements of \mathcal{U} to types implicit.

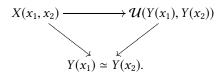
Definition 2.2 (Partial univalence). Let *X* be a type and $Y: X \to \mathcal{U}$ for some universe \mathcal{U} . Let *P* be a propositional family over \mathcal{U} . We say that (X, Y) is *P*-univalent if the restriction of the family *Y* to the subtype $\sum_{x:X} P(Y(x))$ is univalent.

We also say that X is *partially univalent* or *univalent for* P, leaving Y implicit. For $P \equiv type_n$, we say that X is *univalent for n-types*. In particular, for $P \equiv isProp$, we say that X is *univalent for propositions*.

Of particular importance is the case where Y is the identity function. In that case, we say that *the universe* \mathcal{U} *is P*-*univalent*.

Lemma 2.3. Let the universe \mathcal{U} be *P*-univalent. Then $Y: X \rightarrow \mathcal{U}$ is *P*-univalent exactly if its restriction to x: X with P(Y(x)) is an embedding.

Proof. Let $x_1, x_2 : X$ with $P(Y(x_1))$ and $P(Y(x_2))$. Consider the commuting diagram



Since \mathcal{U} is *P*-univalent, the right map is an equivalence. By 2-out-of-3, the left map is invertible exactly if the top map is invertible. Quantifying over x_1, x_2 , we obtain the claim. \Box

2.1 Partially Univalent Type V of (Small) Sets

Let now $\mathcal{U}^{\leq 0}$ be a univalent universe of sets, meaning its elements decode to 0-truncated types. We wish to define a "closed" 0-truncated universe V with a decoding function $\mathsf{El}_V : V \to \mathcal{U}^{\leq 0}$ that is univalent for propositions in the sense of Definition 2.2. We illustrate the essential features of our construction by requiring that:

- V contains codes for a fixed family N : M → U^{≤0} of elements of U^{≤0} where M is a set.
- *V* is closed under Π -types (assuming that $\mathcal{U}^{\leq 0}$ is),

The former family can for example include codes for the empty type or the type of Booleans.

Closed universes are typically defined by induction-recursion, simultaneously defining the type V and the function $\mathsf{El}_V : V \to \mathcal{U}^{\leq 0}$. To model the above closure conditions, one takes:

- given m : M, a constructor $\overline{N}(m) : V$ and a clause $El_V(\overline{N}(m)) \equiv N(m)$.
- given \overline{A} : V and \overline{B} : $\mathsf{El}_V(\overline{A}) \to V$, a constructor $\overline{\Pi}(\overline{A},\overline{B}): V$ and a clause

$$\mathsf{El}_V(\overline{\Pi}(\overline{A},\overline{B})) \equiv \prod_{(a:\mathsf{El}_V(\overline{A}))} \mathsf{El}_V(\overline{B}(a)),$$

In order to make *V* univalent for propositions, one could imagine turning this into a *higher inductive-recursive* definition. Given $\overline{A}_i : V$ with $\mathsf{El}_V(\overline{A}_i)$ a proposition for $i \in \{0, 1\}$ and an equality $p : \mathcal{U}^{\leq 0}(\mathsf{El}_V(\overline{A}_0), \mathsf{El}_V(\overline{A}_1))$, one would add a path constructor $\mathsf{ua}(e) : V(\overline{A}_0, \overline{A}_1)$ with a clause giving an identification of the action of El_V on the path $\mathsf{ua}(e)$ with p.¹ This is the right idea, but there are problems.

- While syntax and semantics of higher inductive types have been analyzed to a certain extent [21, 26] this analysis does not yet extend to the case of inductionrecursion. As such, the rules for higher inductive-recursive types have not yet been established and none of the known models of homotopy type theory have been shown to admit them.
- Induction-recursion is known to increase the prooftheoretic strength of the type theory over just (indexed)

induction. Thus, we do not wish to assume it in our ambient type theory.

Given a type *I*, recall that types *X* with a map $X \to I$ are equivalent to families over *I*: in the forward direction, one takes fibers; in the backward direction, one takes the total type. Exploiting this correspondence, the above inductive-recursive definition of *V* (without path constructor for partial univalence) can be turned into an *indexed inductive* definition of a family inV over $\mathcal{U}^{\leq 0}$. The translation of the constructors for Π -types and *M* is given in (i) and (ii) of Definition 2.4. The path constructor for partial univalence corresponds to the following: given propositions $A_i : \mathcal{U}^{\leq 0}$ with $w_{A_i} : inV(A_i)$ for $i \in \{0, 1\}$ and an equality $p : \mathcal{U}^{\leq 0}(A_0, A_1)$, we have a path ua(*e*) in the family inV betwen w_{A_0} and w_{A_1} over *p*. We contract the path *p* with one of its endpoints and arrive at the definition below.

Definition 2.4. The family inV over $\mathcal{U}^{\leq 0}$ is defined as the following higher indexed inductive type:

- (i) given m : M, a constructor $w_N(m) : inV(N(m))$,
- (ii) given w_A : inV(A) and $w_B(a)$: inV(B(a)) for a : A(with implicit $A : \mathcal{U}^{\leq 0}$ and $B : A \to \mathcal{U}^{\leq 0}$), a constructor $w_{\Pi}(w_A, w_B)$: inV($\prod_{(a:A)} B(a)$),
- (iii) given a proposition $X : \mathcal{U}^{\leq 0}$ with $w_0, w_1 : inV(X)$, a path constructor $ua(w_0, w_1) : w_0 = w_1$.

We recover *V* as the total type $V = \sum_{(X:\mathcal{U}^{\leq 0})} inV(X)$, with El_V given by the first projection.

We may regard inV as a *conditionally* or *partially propositionally truncated* indexed inductive type (see Appendix B). In this form, it becomes clear that the constructor ua indeed suffices for partial univalence and does not introduce coherence problems: it exactly enforces that the restriction of the family inV to elements decoding to propositions is valued in propositions.

Lemma 2.5. The type V with $\mathsf{El}_V : V \to \mathcal{U}^{\leq 0}$ is univalent for propositions.

Proof. Using Lemma 2.3, we have to show that the restriction of $\mathsf{El}_V \colon V \to \mathcal{U}^{\leq 0}$ to $\overline{X} \colon V$ with $\mathsf{El}_V(\overline{X})$ a proposition is an embedding. Unfolding to inV, this says that inV(X) is propositional for $X \colon \mathcal{U}^{\leq 0}$ a proposition. This is exactly enforced by the path constructor (iii) in Definition 2.4.

It remains to show that *V* is 0-truncated. For this, we adapt the encode-decode method to characterize the dependent equalities in inV over an equality in $\mathcal{U}^{\leq 0}$.

2.2 Dependent Equalities in inV

In the following, we make use of (homotopy) pushouts. Recall [34, Section 6.8] that the *pushout* of a span $f: A \rightarrow B$ and $g: A \rightarrow C$ of types is the (non-recursive) higher inductive type $B +_A C$ with points constructors $inl(b) : B +_A C$ for b : B and $inr(c) : B +_A C$ for c : C and path constructor glue(a) : ($B +_A C$)(inl(f(b)), inr(q(c))) for a : A.

¹ In this particular case, the clause of El_V for the path constructor amounts to nothing as it is an identification in a propositional type.

As in [31, Lecture 13], we do not require judgmental β laws even for point constructors. Thus, pushout types are simply a particular choice of pushout squares, (homotopy) initial cocones under the span $B \leftarrow A \rightarrow C$. We refer to Appendix A for some key properties of pushouts used in our development.

An important special case is the *join* $X \star Y$ of types X and Y, the pushout of X and Y under $X \times Y$. For a proposition P, the operation $P \star -$ is also known as the *closed modality* associated with P [32, Example 1.8]. We will only use the join in this form. Recall that $X \star Y$ is contractible if X or Y is contractible. In particular, $P \star X$ is contractible if P holds.

Problem 2.6. Given an equality $p: \mathcal{U}^{\leq 0}(X_0, X_1)$ and $w_i :$ inV (X_i) for $i \in \{0, 1\}$, we wish to define:

- a type Eq_{inV}(p, w₀, w₁) of codes of equalities over p between w₀ and w₁
- such that Eq_{inV}(p, w₀, w₁) is contractible if X₀ (or equivalently X₁) is a proposition.

Construction. By univalence for contractible types, the type of contractible types is contractible. Thus, the goal is contractible if X_0 or X_1 is a proposition.

We perform double induction, first on w_0 : $inV(X_0)$ and then on w_1 : $inV(X_1)$. In all path constructor cases, we know that X_0 or X_1 is a proposition. By the above, the goal becomes an equality in a contractible type, so there is nothing to show.

In all point constructor cases, we define

 $Eq_{inV}(u, w_0, w_1) \equiv_{def} isProp(X_0) \star E$

where *E* is an abbreviation for an expression that varies depending on the case. Note that this makes $Eq_{inV}(u, w_0, w_1)$ contractible when X_0 is a proposition. The expression *E* codes structural equality of the top-level constructors.

- For $w_0 \equiv w_N(m_0)$ and $w_1 \equiv w_N(m_1)$, we let *E* consist of pairs (p_m, c) where $p_m : M(m_0, m_1)$ and *c* is a proof that $p : \mathcal{U}^{\leq 0}(N(m_0), N(m_1))$ is equal to the action of *M* on p_m .
- For $w_i \equiv w_{\Pi}(w_{A_i}, w_{B_i})$ with w_{A_i} : inV(A_i) and $w_{B_i}(a_i)$: inV($B_i(a_i)$) for $a_i : A_i$, all for $i \in \{0, 1\}$, we let *E* consist of tuples (p_A, e_A, p_B, e_B, c) where:
 - $-p_A: \mathcal{U}^{\leq 0}(A_0, A_1) \text{ with } e_A: Eq_{inV}(p_A, w_{A_0}, w_{A_1}),$
 - for $a_0 : A_0, a_1 : A_1$, and a dependent equality p_a over p_A between a_0 and a_1 , we have $p_B : \mathcal{U}^{\leq 0}(B_0(a_0), B_1(a_1))$ with $e_B : \text{Eq}_{inV}(p_B, w_{B_0}(a_0), w_{B_1}(a_1))$.
 - c witnesses that

 $p: \mathcal{U}^{\leq 0}(\prod_{(a_0:A_0)} B_0(a_0), \prod_{(a_1:A_1)} B_1(a_1))$

is equal to the action of the type forming operation Π on p_A and p_B .

• In the remaining "mixed" cases, we let *E* be empty. \Box

Proposition 2.7. Given $p : \mathcal{U}^{\leq 0}(X_0, X_1)$ and $w_i : inV(X_i)$ for $i \in \{0, 1\}$, there is an equivalence between dependent equalities in inV over p between w_0 and w_1 and $Eq_{inV}(p, w_0, w_1)$. *Proof.* For the purpose of this proof, it will be convenient to work with a different, but equivalent definition of the expression *E* in the construction of Problem 2.6 in the case $w_i \equiv w_{\Pi}(w_{A_i}, w_{B_i})$ with $w_{A_i} : \text{inV}(A_i)$ and $w_{B_i}(a_i) : \text{inV}(B_i(a_i))$ for $a_i : A_i$, all for $i \in \{0, 1\}$. Namely, we let *E* consist of pairs (q, r) as follows.

- The component q is an equality $(A_0, B_0) = (A_1, B_1)$ in the dependent sum $\sum_{(A: \mathcal{U}^{\leq 0})} A \to \mathcal{U}^{\leq 0}$.
- Inducting on the equality q, we may suppose $A \equiv_{def} A_0 \equiv A_1$ and $B \equiv_{def} B_0 \equiv B_1$. The component r is then a triple (e_A, e_B, c) where
 - $e_A : \mathsf{Eq}_{\mathsf{inV}}(\mathsf{refl}_A, w_{A_0}, w_{A_1}),$
 - $e_B(a) : \mathsf{Eq}_{\mathsf{inV}}(\mathsf{refl}_{B(a)}, w_{B_0}(a), w_{B_1}(a)) \text{ for } a : A,$ $- c : p = \mathsf{refl}.$

The equivalence between this choice of E and the previous one is a staightforward consequence of structural equivalences, splitting up the equality q into components for A and B and distributing them over the components of r.

We follow the encode-decode method as described in Subappendix B.3. To define

$$w_0 =_{\mathsf{inV}(p)} w_1 \xrightarrow{\mathsf{encode}_{p, w_0, w_1}} \mathsf{Eq}_{\mathsf{inV}}(p, w_0, w_1),$$

we use equality induction on p and the argument, reducing the goal to encode'(w) : Eq_{inV}(refl_X, w, w) for w : inV(X). We induct on x : inV(X).

• For $w \equiv w_N(m)$: inV(N(m)), we take

 $encode'(x) \equiv inr(refl_m, refl).$

• For $w \equiv w_{\Pi}(w_A, w_B)$: inV $(\prod_{(a:A)} B(a))$, we take

 $encode'(x) \equiv inr$

- $(refl_{(A,B)}, (encode'(w_A), \lambda a. encode'(w_B(a)), refl)).$
- In the path constructor case, we have that *X* is a proposition. Then the goal is a dependent equality in a contractible type.

We now show $\operatorname{encode}_{p, w_0, w_1}^{-1}(e)$ for $e : \operatorname{Eq}_{\operatorname{inV}}(p, w_0, w_1)$. We use double induction on w_0 and w_1 . In all path constructor cases, we know that X_0 or X_1 is a proposition (hence both are). Thus, both source and target of $\operatorname{encode}_{p, w_0, w_1}$ are contractible, so the goal becomes contractible. In all point constructor cases, we have $e : \operatorname{isProp}(X_0) \star E$ where E depends on the particular case. We induct on e. In the case for inl or glue, we have isProp(X_0), and the goal becomes contractible. For $e \equiv \operatorname{inr}(z)$, we proceed with z according to the point constructor case for w_0 and w_1 .

• For $w_0 \equiv w_N(m_0)$ and $w_1 \equiv w_N(m_1)$, we have $z = (p_m, c)$. By equality induction on $p_m : M(m_0, m_1)$, we may suppose $m \equiv_{def} m_0 \equiv m_1$ and $p_m \equiv \text{refl}_m$. By equality induction on *c*, we may then suppose $p \equiv \text{refl}$

and $c \equiv$ refl. We have

е

$$\equiv$$
 inr(refl_m, refl)

$$\equiv$$
 encode'($w_N(m)$)

 \equiv encode_{refl_{N(m)}, w_N(m), w_N(m)(refl),}

showing encode $p_{p, w_0, w_1}^{-1}(e)$.

• For $w_i \equiv w_{\Pi}(w_{A_i}, w_{B_i})$ with $w_{A_i} : \text{inV}(A_i)$ and $w_{B_i}(a_i) :$ inV $(B_i(a_i))$ for $a_i : A_i$, all for $i \in \{0, 1\}$, we have z = (q, r) as described at the beginning of this proof. By equality induction on $q : (A_0, B_0) = (A_1, B_1)$, we may suppose $A \equiv_{\text{def}} A_0 \equiv A_1, B \equiv_{\text{def}} B_0 \equiv B_1$, and $q \equiv_{\text{def}}$ refl. Then $r = (e_A, e_B, c)$. By equality induction c, we may suppose that $p \equiv$ refl and $c \equiv$ refl. By induction hypothesis, we have

$$encode_{refl_{A}, w_{A_{0}}, w_{A_{1}}}^{-1}(e_{A}),$$

 $encode_{refl_{B(a)}, w_{B_{0}}(a), w_{B_{1}}(a)}^{-1}(e_{B}(a))$ for $a : A$

By equality induction and function extensionality, we may thus suppose that

$$e_A \equiv \text{encode}_{\text{refl}_A, w_{A_0}, w_{A_1}}(q_A),$$

$$e_B \equiv \lambda a. \text{encode}_{\text{refl}_{B(a)}, w_{B_0}(a), w_{B_1}(a)}(q_B(a))$$

for some $q_A : w_{A_0} = w_{A_1}$ and $q_B(a) : w_{B_0}(a) = w_{B_1}(a)$ for a : A. By equality induction on q_A and q_B (after using function extensionality), we may suppose that $w_A \equiv_{def} w_{A_0} \equiv w_{A_1}$ and $q_A \equiv$ refl as well as $w_B \equiv_{def}$ $w_{B_0} \equiv w_{B_1}$ and $q_B \equiv \lambda a$. refl. Now we have

$$e \equiv inr(refl_{(A,B)}, (encode'(w_A), \lambda a. encode'(w_B(a)), refl))$$

$$\equiv$$
 encode' $(w_{\Pi}(w_A, w_B))$

 $\equiv \text{encode}_{\text{refl}_{\prod_{(a:A)} B(a)}, w_{\Pi}(w_A, w_B), w_{\Pi}(w_A, w_B)}(\text{refl}),$ showing $\text{encode}_{p, w_0, w_1}^{-1}(e).$

• In all "mixed" cases, we have
$$z : 0$$
.

For readers concerned with the length of the above argument, we note the following. In Section 3, we will motivate abstraction that will allow us to reorganize the above argument into smaller, more general pieces.

2.3 V is a set

From our characterization of dependent equality in inV, we obtain a corresponding characterization of equality in V. Given $\overline{X}_i \equiv (X_i, w_i) : V$ for $i \in \{0, 1\}$, we define

$$\mathsf{Eq}_{V}(\overline{X}_{0},\overline{X}_{1}) \equiv_{\mathsf{def}} \sum_{p:\mathcal{U}^{\leq 0}(X_{0},X_{1})} \mathsf{Eq}_{\mathsf{inV}}(p,w_{0},w_{1}).$$

Corollary 2.8. For $\overline{X}_0, \overline{X}_1 : V$, we have $V(\overline{X}_0, \overline{X}_1) \simeq \mathsf{Eq}_V(\overline{X}_0, \overline{X}_1).$

Proof. Equality types in the dependent sum $V \equiv \sum_{(X:\mathcal{U}^{\leq 0})} \text{inV}(X)$ are dependent sums of an equality in $\mathcal{U}^{\leq 0}$ and a dependent equality over it. Thus, the claim is a consequence of Proposition 2.7.

Proposition 2.9. The type V is 0-truncated.

Proof. Given $\overline{X}_i \equiv (X_i, w_i) : V$ for $i \in \{0, 1\}$, we wish to show $V(\overline{X}_0, \overline{X}_1)$ propositional. By Corollary 2.8, this amounts to showing $Eq_V(\overline{X}_0, \overline{X}_1)$ propositional. This we show by double induction, first on $w_0 : inV(X_0)$ and then on $w_1 : inV(X_1)$. Since the goal is propositional, there is nothing to show in the path constructor cases.

In all point constructor cases, we have

Ε

$$q_V(\overline{X}_0, \overline{X}_1) = \sum_{p: \mathcal{U}^{\leq 0}(X_0, X_1)} \operatorname{isProp}(X_0) \star E(p)$$
(1)

where E(p) is as in the construction for Problem 2.6, abbreviating an expression depending on the point constructor case (for clarity, we have made the dependency on p explicit). By definition, this join forms a pushout square

The dependent sum over a fixed type preserves pushout squares in its remaining argument (abstractly, because it is a higher functor left adjoint to weakening). From (1) and (2), we thus obtain the following pushout square:

Since $\mathcal{U}^{\leq 0}$ is univalent, the type $\mathcal{U}^{\leq 0}(X_0, X_1)$ is propositional if is Prop (X_0) . From this, we see that the span in (3) is a (homotopy) product span. By invariance of pushouts under equivalence, it follows that $\operatorname{Eq}_V(\overline{X}_0, \overline{X}_1)$ is equivalent to the join

$$\left(\mathcal{U}^{\leq 0}(X_0, X_1) \times \mathsf{isProp}(X_0)\right) \star \left(\sum_{p: \mathcal{U}^{\leq 0}(X_0, X_1)} E(p)\right).$$
(4)

We now apply Lemma A.9: to show that this join is propositional, it suffices to show that each of its factors is propositional.² Since isProp(X_0) is propositional and $\mathcal{U}^{\leq 0}(X_0, X_1)$ is propositional if isProp(X_0), their product is propositional.

It remains to show that $T \equiv_{def} \sum_{(p:\mathcal{U}^{\leq 0}(X_0, X_1))} E(p)$ is a proposition. For this, we argue according to the current point constructor case, recalling the corresponding definition of E(p) from Problem 2.6.

• In the case $w_0 \equiv w_N(m_0)$ and $w_1 \equiv w_N(m_1)$, we have

$$T \equiv \sum_{\substack{(p:\mathcal{U}^{\leq 0}(X_0, X_1)) \ (p_m: \mathcal{M}(m_0, m_1))}} \sum_{\substack{(p = ap_N(p_m)) \\ \simeq \mathcal{M}(m_0, m_1),}} (p = ap_N(p_m))$$

²Alternatively, we could appeal to the fact that closed modalities are left exact, hence preserve truncation levels [32].

a proposition since M was assumed a set.

- In the case $w_i \equiv w_{\Pi}(w_{A_i}, w_{B_i})$ with $w_{A_i} : \text{inV}(A_i)$ and $w_{B_i}(a_i) : \text{inV}(B_i(a_i))$ for $a_i : A_i$ for $i \in \{0, 1\}$, recall that *T* consists of tuples $(p, p_A, e_A, p_B, e_B, c)$ with types as in the construction of Problem 2.6. We contract the equality *c* with its endpoint *p*. What remains is equivalent to the dependent sum of:
 - (p_A, e_A) : Eq_V $(\overline{A}_0, \overline{A}_1)$ where $\overline{A}_i \equiv_{def} (A_i, w_i)$,
 - for $a_0 : A_0, a_1 : A_1$, and a dependent equality p_a over p_A between a_0 and a_1 :

$$(p_B, e_B) : \operatorname{Eq}_V(\overline{B}_0(a_0), \overline{B}_1(a_1))$$

where $\overline{B}_i(a_i) \equiv_{\text{def}} (B_i(a_i), w_i(a_i))$. Both $\text{Eq}_V(\overline{A}_0, \overline{A}_1)$ and $\text{Eq}_V(\overline{B}_0(a_0), \overline{B}_1(a_1))$ (the latter for all a_0, a_1) are (n - 1)-truncated by induction hypothesis. The claim now follows by closure of (n - 1)-truncated types under dependent sums and dependent products (with arbitrary domain).

• In the remaining, "mixed" cases, we have

$$T \equiv \sum_{p: \mathcal{U}^{\leq 0}(X_0, X_1)} \bot \simeq \bot,$$

which is propositional.

3 *n*-truncated Partially Univalent Universes of *n*-types

To obtain the main result of this paper, we need to generalize the constructions of the previous section to a partially univalent *n*-truncated universe of *n*-types rather than sets, and to a universe closed under more type formers. For the sake of generality, we will also build a universe that is *P*-univalent for an arbitrary proposition *P*, although the *n*-truncatedness result will need P(X) to imply type_(*n*-1)(*X*).

This section only makes use of univalence for (n - 1)-types.

3.1 Indexed Containers and Preservation of Truncation

To abstract from a particular choice of type formers, we will parametrize our universe by a signature of them represented by an indexed container [1], as is done for indexed W-types.

We recall here the precise definitions of indexed container and its extension that we will use in the rest of the paper.

Definition 3.1 (Indexed container). Given a type *I*, an *I*indexed container is a pair (*S*, *Pos*) of a type family *S* over *I* and a type family *Pos* over $\sum_{(i:I)} S(i) \times I$.

Definition 3.2 (Extension of a container). Let (S, Pos) be an *I*-indexed container. Its extension $Ext_{S,Pos}$ takes a family *F* over *I* and produces another:

$$\mathsf{Ext}_{S, \mathit{Pos}}(F, i) = \sum_{(s:S(i))} \prod_{(j:I)} \mathit{Pos}(i, s, j) \to F(j)$$

Given (s, t) : Ext_{S.Pos}(F, i) we will write $t_Y(p)$ for t(Y)(p).

In the universe construction we will use a $\mathcal{U}^{\leq n}$ -indexed container, here we demonstrate by example that they not only cover the type formers considered in Section 2, but also ones with a more complex signature like (truncated) pushouts.

Example 3.3 (Nullary type formers). Given a fixed family of types $N : M \to \mathcal{U}^{\leq n}$ we define a container with empty positions:

$$S(X) = \sum_{(m:M)} (X = N(m))$$

$$Pos(_,_,_) = \bot$$

Example 3.4 (Π -types). The signature for Π -types can be represented by a $\mathcal{U}^{\leq n}$ -indexed container where both *S* and *Pos* are given by indexed inductive types with constructors:

• given $A : \mathcal{U}^{\leq n}$ and $B : A \to \mathcal{U}^{\leq n}$ a constructor $\pi(A, B) : S(\prod_{(x:A)} B(x)).$

and with $s \equiv \pi(A, B)$ and $X \equiv \prod_{(a:A)} B(a)$:

- a constructor $pos_A : Pos(X, s, A)$
- given a : A a constructor $pos_B : Pos(X, s, B(a))$

Example 3.5 (Truncated pushouts). Pushouts truncated to be *n*-types can also be represented as a $\mathcal{U}^{\leq n}$ -indexed container:

- given $A_i : \mathcal{U}^{\leq n}$ for $i \in \{0, 1, 2\}$ and $f : A_0 \to A_1$ and $g : A_0 \to A_2$ a constructor $po(f, g) : S(A_1 + ^n_{A_0} A_2)$,
- for each $i \in \{0, 1, 2\}$ a constructor $pos_i : Pos(A_1 +_{A_0}^n A_2, po(f, g), A_i).$

More generally, this works for arbitrary HITs with an additional constructor ensuring *n*-truncatedness.

To establish the *n*-truncatedness of the universe we will need to know that the extension of the container $\text{Ext}_{S,P}(F, i)$ preserves the truncation level of the family *F*. We cannot however just ask for type_n($\sum_{(i:I)}$.Ext_{S,P}(*F*, *i*)) to hold whenever type_n($\sum_{((i:I))} F(i)$) holds, as the latter would already be the whole result when $F \equiv \text{inV}$. We extract then the following condition from what is needed during the induction in the proof of Theorem 3.13.

Definition 3.6 (Retaining *n*-truncatedness). An *I*-indexed container (*S*, *Pos*) *retains n*-truncation, if for any family *F* over *I*, any $i_b : I$ and element of the extension $(s_b, t_b) :$ Ext_{*S*, *Pos*(*F*, i_b) for $b \in \{0, 1\}$ we have that}

 $\prod_{\substack{(j_0, j_1:I) \\ \text{type}_{(n-1)}(\sum_{\substack{(q:j_0=j_1) \\ (p_0:p_0) \\ (p_0) = F(q) \\ (p_1, p_1))}} } \prod_{\substack{(p_0:Pos(i_0, s_0, j_0), p_1:Pos(i_1, s_1, j_1)) \\ (p_0:Pos(i_0, s_0, j_0), p_1:Pos(i_1, s_1, j_1)) \\ (p_0:Pos(i_1, s_1, j_1), p_2:Pos(i_1, s_1, j_1)) \\ (p$

$$\mathsf{type}_{(n-1)}((i_0, s_0, t_0) =_{\sum_{(i:1)} \mathsf{Ext}_{S, \mathsf{Pos}}(F, i)} (i_1, s_1, t_1))$$

Example 3.7 (Signatures retaining *n*-truncatedness). Examples 3.3 to 3.5 all retain *n*-truncatedness. The case of nullary type formers is trivial. As mentioned the case for Π -types follows the reasoning in Proposition 2.9. For truncated pushouts we can observe that (X, (s, t)) of type $\sum_{(X:\mathcal{U}^{\leq n})} \operatorname{Ext}_{S, Pos}(F, X)$ is equivalent to the following data:

•
$$X: \mathcal{U}^{\leq n}$$

• for $i \in \{0, 1, 2\}$, both $A_i : \mathcal{U}^{\leq n}$ and $w_i : F(A_i)$

•
$$f: A_0 \to A_1$$
 and $g: A_0 \to A_2$

•
$$q: X = A_1 +_{A_0}^n A_2$$

then *X* and *q* form a contractible pair, the types of *f* and *g* are *n*-truncated by construction, so we only have to worry about the (A_i, w_i) pairs. But since the w_i are obtained from *t*, those components are handled by the premise given to us.

Coproducts of such containers also retain *n*-truncatedness as their extension will correspond to the sum of the extensions, which means we can collect multiple type formers into a single *n*-truncatedness preserving indexed container.

3.2 A *P*-univalent *n*-truncated Universe of *n*-truncated Types

Now we have everything in place to provide the final version of our universe

$$\mathsf{V} = \sum_{X: \mathcal{U}^{\leq n}} \mathsf{inV}_{S, \mathit{Pos}}^{n, P}(X)$$

with $El_V : V \to \mathcal{U}^{\leq n}$ given by first projection. We will often omit the sub- and sup- scripts on inV as they will be clear from context.

The family inV is defined as follows. In analogy with the indexed W-type $W_{S,Pos}$, which one would use for an ordinary closed universe, we use the *P*-propositional indexed W-type inV $\equiv_{def} W_{S,Pos}^{P}$ (Definition B.3). The theory of partially propositional indexed W-types is developed in Appendix B. For convenience, we give here the explicit definition as an indexed higher inductive type.

Definition 3.8. Given a $\mathcal{U}^{\leq n}$ -indexed container (*S*, *Pos*), the family inV over $\mathcal{U}^{\leq n}$ is defined as the higher inductive type generated by the following constructors:

- (i) given $c : Ext_{S, Pos}(inV, X)$, a constructor tcon(c) : inV(X),
- (ii) given $X : \mathcal{U}^{\leq n}$ with P(X), and $w_0, w_1 : inV(X)$, a path constructor $ua(w_0, w_1) : w_0 =_{inV(X)} w_1$.

Lemma 2.5 generalizes to the new setting.

Lemma 3.9. Let (S, Pos) be a $\mathcal{U}^{\leq n}$ -indexed container, and P a family of propositions over $\mathcal{U}^{\leq n}$. The type V with $El_V : V \rightarrow \mathcal{U}^{\leq n}$ is P-univalent. \Box

We unfold here the definition of codes for equality in $W^P_{S Pos}$ of Subappendix B.4.

Problem 3.10. Given an equality $p: \mathcal{U}^{\leq n}(X_0, X_1)$ and $w_i :$ inV (X_i) for $i \in \{0, 1\}$, we define:

- a type Eq_{inV}(p, w₀, w₁) of codes of equalities between w₀ and w₁ over p, as in Figure 1.
- such that $Eq_{inV}(p, w_0, w_1)$ is contractible if $P(X_0)$.

Construction. The definition proceeds by double induction on w_0 and w_1 , defining the pair of $Eq_{inV}(p, w_0, w_1)$ and its conditional contractibility in one go. Given $P(X_0)$ the case when both w_i are built with tcon is contractible because it's a join with an inhabited proposition. When either w_0 or w_1 is built by ua we again have by univalence that the type of contractible types is contractible. $\hfill \Box$

From Subappendix B.4, we have the following result.

Proposition 3.11. Given $p : \mathcal{U}^{\leq n}(X_0, X_1)$ and $w_i : inV(X_i)$ for $i \in \{0, 1\}$, there is an equivalence

$$w_0 =_{\mathsf{inV}(p)} w_1 \simeq \mathsf{Eq}_{\mathsf{inV}}(p, w_0, w_1). \qquad \Box$$

As in Section 2 we define

$$\mathsf{Eq}_{\mathsf{V}}(p,\overline{X}_0,\overline{X}_1) \equiv_{\mathsf{def}} \sum_{p:\mathcal{U}^{\leq n}(X_0,X_1)} \mathsf{Eq}_{\mathsf{inV}}(p,w_0,w_1).$$

for $\overline{X}_i \equiv (X_i, w_i)$ for $i \in \{0, 1\}$ and derive its equivalence with equality in V.

Corollary 3.12. For $\overline{X}_0, \overline{X}_1 : V$, we have

$$\mathsf{V}(\overline{X}_0, \overline{X}_1) \simeq \mathsf{Eq}_{\mathsf{V}}(\overline{X}_0, \overline{X}_1).$$

3.2.1 V is *n*-truncated.

Theorem 3.13. Let (S, Pos) be an *n*-truncatedness retaining container. If P(X) implies type_{*n*-1}(X) then V is *n*-truncated.

Proof. Given $\overline{X}_i \equiv (X_i, w_i) : V$ for $i \in \{0, 1\}$ we proceed by induction on w_0 and w_1 to prove $V(\overline{X}_0, \overline{X}_1)$ is (n - 1)truncated. By Corollary 3.12 and the same reasoning as in the proof of Proposition 2.9, we have to concern ourselves only with the following pushout square³:

where $E(p) = \mathsf{Eq}'_{inV}(p, (s_0, t_0), (s_1, t_1))$. By Proposition A.11, it is enough to show the top right and bottom left corners are (n-1)-truncated to conclude that $\mathsf{Eq}_V(\overline{X}_0, \overline{X}_1)$ is as well. $\sum_{(p:\mathcal{U}^{\leq n}(X_0, X_1))} P(X_0)$ is (n-1)-truncated because $P(X_0)$ is a proposition and implies $\mathsf{type}_{n-1}(X_0)$, so that $\mathcal{U}^{\leq n}(X_0, X_1)$ is (n-1)-truncated by univalence. $\sum_{(p:\mathcal{U}^{\leq n}(X_0, X_1))} E(p)$ is equivalent to $(X_0, (s_0, t_0)) = (X_1, (s_1, t_1))$ by Proposition 3.11, so we can conclude its (n-1)-trucatedness by using that (S, Pos)retains *n*-truncatedness, because its premise is satisfied by the induction hypothesis.

As the special case where P is constantly false, we obtain the folklore construction of 0-truncated "closed" universes.

Corollary 3.14. Let (S, Pos) be a 0-truncatedness retaining container. If $P = \lambda X \perp$ then \forall is 0-truncated.

 $[\]overline{}^{3}$ a variant of (3)

$$\mathsf{Eq}_{\mathsf{inV}}'(q, (s_0, t_0), (s_1, t_1)) \equiv_{\mathsf{def}} \Sigma (q_s : s_0 =_{S(q)} s_1). \\ (e_t : \prod_{(Y:\mathcal{U}^{\leq n})} \prod_{(p_0, p_1)} p_0 =_{\operatorname{Pos}(q, q_s, Y)} p_1 \to \mathsf{Eq}_{\mathsf{inV}}(\mathsf{refl}, t_0_Y(p_0), t_1_Y(p_1)))$$

 $\mathsf{Eq}_{\mathsf{inV}}(q:X_0 = X_1, w_0, w_1) \equiv_{\mathsf{def}} \begin{cases} P(X_0) \star \mathsf{Eq}'_{\mathsf{inV}}(q, c_0, c_1) & \text{if } w_0 \equiv \mathsf{tcon}(c_0), w_1 \equiv \mathsf{tcon}(c_1) \\ \text{by contractibility} & \text{if } w_0 \text{ or } w_1 \text{ given by ua}(\ldots) \end{cases}$

Figure 1. Definition of Eq_{inV}.

4 Models of *n*-truncated Type Theory with Univalence for (n - 1)-types.

In this section, we show that Martin-Löf type theory with function extensionality and the assumption that all types are *n*-truncated is consistent with univalence for (n - 1)-types.

Remark 4.1. The full strength of this statement is realized only with a sufficiently long chain of universes $\mathcal{U}_0, \ldots, \mathcal{U}_k$, one included in the next. For if k < n, it is known [23, Section 6] how to modify a model of homotopy type theory (including univalence, but no higher inductive types) to be *n*-truncated by restricting types of "size" *i* (classified by \mathcal{U}_i) to *i*-types (and restricting all types to the *n*-truncated).

For the reason given in the above remark, we consider Martin-Löf type theory to come with an ω -indexed (cumulative) hierarchy of universes $\mathcal{U}_0, \mathcal{U}_1, \ldots$ Alternatively, we could include higher inductive types, which in the presence of univalence for (n-1)-types are still able to produce proper *n*-types.⁴ However, a key point is still for an *n*-truncated universe univalent for (n-1)-types to be able to contain a code for a (smaller) universe of the same kind, and at this point we may as well consider a hierarchy of universes.

We use *categories with families* (cwfs) [15] as our notion of model of dependent type theory. They are models of a generalized algebraic theory [9] (as will all semantic notions considered here). Fixing the underlying category C, we obtain a category of *cwf structures* on C. We refer to a cwf structure on C by its presheaf of types Ty, the presheaf Tm of terms on its category of elements $\int Ty$ left implicit.

Let *T* stand for a choice of type formers, specified by a collection of rules that are generally natural in the context (one way to ensure this naturality is by demanding that these rules be interpretable in presheaves over the category of contexts and substitutions [5, 8]). Type formers can be standard type formers such as dependent sums, dependent products, or identity types, but also "axioms" such as function extensionality. As before, we have categories of *cwfs with type formers T* as well as *cwf structures with type formers T* on a fixed category *C*.

Definition 4.2 ([4, Definition 2.4]). A *cwf hierarchy* Ty *with type formers T* on a category *C* is a sequential diagram

$$Ty_0 \longrightarrow Ty_1 \longrightarrow \dots$$

of cwf structures with type formers T on C.

Note that the *lifting maps* $Ty_i \rightarrow Ty_{i+1}$ preserve type formers *T*. As before, cwf hierarchies with type formers *T* assemble into a category.

Definition 4.3 ([4, Definition 2.5]). A model of Martin-Löf type theory with type formers *T* is a category *C* with a cwf hierarchy Ty with type formers *T* on *C* together with, for each *i*, a global section \mathcal{U}_i with an isomorphism $\mathsf{El}_i : \mathsf{Tm}_{i+1}(\Gamma, \mathcal{U}_i) \simeq$ $\mathsf{Ty}_i(\Gamma)$ natural in $\Gamma \in C$.

In all uses of the above definition, we will implicitly assume that *T* contains at least dependent sums, dependent products, identity types, and finite coproducts. This makes available basic concepts of homotopy type theory such as being *n*-truncated (for an external number *n*). We write $MLTT_T$ for the category such models, and $MLTT_T(C)$ if we wish to fix the underlying category. We say that *C* is *n*-truncated (where $n \ge -2$) if all $A \in Ty_i(\Gamma)$ are *n*-truncated, naturally in $\Gamma \in C$, for all *i*. As a type former (an axiom), we denote it Tr(n).

We call \mathcal{U}_i the *i*-th universe of *C*. Note that, in contrast to our internal reasoning, we explicitly reference the decoding natural transformation El_i . Given a generic property *P* of types (such as being *n*-truncated) forming an internal proposition, we say that *C* satisfies univalence for *P* if the universe \mathcal{U}_i is *P*-univalent for all *i* in the sense of what follows Definition 2.2. We add a subscript UA(*n*) to MLTT_T to indicate restriction to models with univalence for *n*-types.

Let *T* be a collection of type formers. Given $C \in \mathsf{MLTT}_T$ and $i \ge 0$, the type forming operations contained in *T* can be encoded as internal operations on the universe \mathcal{U}_i . Assume that these internal operations restrict to the subuniverse $\mathcal{U}_i^{\le n}$ of *n*-types. For well-behaved *T*, it is possible to find a global \mathcal{U}_i -indexed container *C* of size i + 1 such that for any family *S* over $\mathcal{U}_i^{\le n}$, a lift of the given internal operations on $\mathcal{U}_i^{\le n}$ to $\sum_{(X:\mathcal{U}_i^{\le n})} S(X)$ corresponds to a *C*-algebra structure on *S*. Lastly, assume that *C* retains *n*-truncatedness in the sense of Definition 3.6. If all of this is the case, naturally in *C*, we say that *T* is *n*-benign.

⁴ For example, univalence for 0-types is sufficient to show that the circle S^1 is a proper 1-type. We consider an *n*-type *proper* if it is not an (n-1)-type.

• Unit type:	• Dependent products:	• Empty type:
S = 1, $t(\bullet) = 1,$ Pos(A, x, y) = 0. • Dependent sums:	$S = \sum_{(A:\mathcal{U}^{\leq n})} A \to \mathcal{U}^{\leq n},$ $t(A, B) = \prod_{(a:A)} B(a),$ $Pos(A, B) = 1 + A,$ $s((A, B), inl(\bullet)) = A,$	S = 1, $t(\bullet) = 0,$ Pos(A, x, y) = 0. • Binary coproducts:
• Dependent sums. $S = \sum_{(A:\mathcal{U}^{\leq n})} A \to \mathcal{U}^{\leq n},$ $t(A, B) = \sum_{(a:A)} B(a),$	s((A, B), inr(a)) = B(a). • Identity types: $S = \sum_{(A:\mathcal{U}^{\leq n})} A \times A,$	$S = \mathcal{U}^{\leq n} \times \mathcal{U}^{\leq n},$ $t(A, B) = A + B,$
Pos(A, B) = 1 + A, $s((A, B), inl(\bullet)) = A,$ s((A, B), inr(a)) = B(a).	t(A, x, y) = A(x, y), Pos(A, x, y) = 1, $s((A, x, y), \bullet) = A.$	Pos(A, B) = 1 + 1, $s((A, B), inl(\bullet)) = A,$ $s((A, B), inr(\bullet)) = B.$

Figure 2. $\mathcal{U}^{\leq n}$ -indexed containers for basic type formers. The specifying data is given in a slightly alternate form: a type of shapes *S*, a target function $t: S \to \mathcal{U}_i^{\leq n}$, a family of positions *Pos* over *S*, and a source function $s: \prod_{s:S} Pos(s) \to \mathcal{U}_i^{\leq n}$. This corresponds to a polynomial functor $\mathcal{U}^{\leq n} \leftarrow S \to P \to \mathcal{U}^{\leq n}$ with middle arrow a fibration. The actual indexed container is obtained by taking fibers using the identity type.

Example 4.4. The basic type formers we implicitly require for a model of Martin-Löf type theory are *n*-benign for any $n \ge 0$. The associated $\mathcal{U}_i^{\le n}$ -indexed containers are listed in Figure 2 (with the size index *i* omitted). Retention of *n*truncatedness in the sense of Definition 3.6 follows the scheme of Example 3.7.

Example 4.5. Any type former that only has term forming operations is automatically *n*-benign. In the first place, this applies to axiom-style type formers such as function extensionality.

We are now ready to state the main result.

Theorem 4.6. Let $n \ge 0$ and T be an n-benign choice of type formers, including function extensionality. Let (C, Ty) be a model of Martin-Löf type theory with type formers T that is univalent for (n-1)-types. Then there is an n-truncated model Ty' of Martin-Löf type theory with type formers T on C that is univalent for (n-1)-types. Furthermore, there is a morphism Ty' \rightarrow Ty of cwf hierarchies with type formers T on C.

Proof. Given a cwf structure Ty on a category *C*, note that a further cwf structure Ty' together with a morphism Ty' \rightarrow Ty corresponds up to isomorphism to just a presheaf Ty' of types with a natural transformation Ty' \rightarrow Ty.⁵ The terms of Ty' are inherited (up to isomorphism) from those of Ty since terms correspond to sections of context projections and Ty' \rightarrow Ty should preserve context extension. Abstractly speaking, the forgetful functor from cwf structures on *C* to discrete fibrations on *C* is itself a discrete fibration.

Let us further assume that Ty implements some type type formers *T*. To interpret *T* in Ty' such that Ty' \rightarrow Ty preserves *T*, we only have to interpret the actual type forming operations of *T* in Ty' such that they are preserved by Ty' \rightarrow Ty; the term forming operations of *T* will then be uniquely inherited from Ty.

Let us now return to the situation of Theorem 4.6. The type forming operations of the type formers T in (C, Ty_i) can be encoded as internal operations on the universe \mathcal{U}_i .⁶ Since T is n-benign, these internal operations further restrict to the subuniverse $\mathcal{U}_i^{\leq n}$ of n-types. We now wish to define a global type V_i of size i + 1 with a map $V_i \to \mathcal{U}_i^{\leq n}$. Restricting El_i along this map, we can see V_i as a universe. Defining $\mathsf{Ty}_i'(\Gamma) = \mathsf{Tm}(\Gamma, V_i)$ with $\mathsf{Ty}_i' \to \mathsf{Ty}_i$ induced by $V_i \to \mathcal{U}_i^{\leq n}$, we then obtain the cwf structure Ty_i' with a map $\mathsf{Ty}_i' \to \mathsf{Ty}_i$. Interpreting T in Ty_i' compatible with Ty_i will follow from a (strict) lift of the internal type formation operations from $\mathcal{U}_i^{\leq n}$ to V_i . Finally, everything needs to be natural in $i \in \omega$.

Let us start with the base i = 0. We will define $V_0 \equiv_{def} \sum_{(X:\mathcal{U}_0^{\leq n})} inV_0$ for a family inV_0 over $\mathcal{U}_0^{\leq n}$, with $V_0 \to \mathcal{U}_0^{\leq n}$ the first projection. Using that T is *n*-benign, we have a $\mathcal{U}_0^{\leq n}$ -indexed container C_0 such that a lift of the internal formations operations from $\mathcal{U}_0^{\leq n}$ to V_0 corresponds to a C_0 -algebra structure on inV_0 . We now follow Section 3 for the construction of inV_0 from C_0 ; this means inV_0 is the type $_{n-1}^{-1}$ -propositional indexed W-type $W_{C_0}^{type_{n-1}}$ as per Subappendix B.5. In particular, we obtain a C_0 -algebra structure on inV_0 . Note

⁵This is immediate when switching from cwfs to the equivalent notion of *categories with attributes*.

⁶Note that this is only a bijective correspondence if we have the judgmental η -law for dependent products, but this is not required here.

that V_0 is univalent for (n - 1)-types by Lemma 3.9 and *n*-truncated by Theorem 3.13.

For general *i*, we let C'_i be the coproduct of the $\mathcal{U}_i^{\leq n}$ indexed container C_i given from *T* being *n*-benign with the indexed container with shapes $0 \leq j < i$, with indexing of *j* being V_j (lifted to Ty_i), and no positions. We then define inV_i from C'_i as before. This guarantees that there are codes for the universes below *i* in V_i .

To make Ty' into a cwf hierarchy and Ty' \rightarrow Ty into a morphism of cwf hierachies, we need to construct, for every $i \ge 0$, a dotted morphism making the naturality square

$$\begin{array}{c} \mathsf{Ty}'_i \longrightarrow \mathsf{Ty}_i \\ \downarrow \\ \mathsf{Ty}'_{i+1} \longrightarrow \mathsf{Ty}_{i+1} \end{array}$$

of presheaves of types commute. This amounts to defining internal $V_i \rightarrow V_{i+1}$ making the square

commute strictly. In turns, this corresponds to an internal function

$$\operatorname{inV}_{i}(X) \to \operatorname{inV}_{i+1}(\operatorname{lift}(X))$$
 (6)

for $X : \mathcal{U}_i^{\leq n}$. We define this by recursion for inV_i , noting that inV_{i+1} restricted along lift carries a C'_i -algebra structure, forgetting the code for the *i*-th universe in its C'_{i+1} -algebra structure.

It remains to check that the map $Ty'_i \rightarrow Ty'_{o+1}$ respects the type forming operations of *T*. This follows from (6) commuting strictly with C_i -algebra structures. This follows from the judgmental β -law of the higher inductive family inV_i.⁷

It remains to check that Ty' has universes as required by Definition 4.3. Indeed, the *i*-th universe \mathcal{U}'_i is simply given by V_i itself, with El'_i the identity isomorphism. \Box

Corollary 4.7. Relative to Martin-Löf type theory with function extensionality and univalence for (n - 1)-types (and any further n-benign type formers), if the addition of pushouts and propositionally truncated indexed W-types is consistent, then it is consistent to assume that all types are n-truncated.

Proof. Given a model for the former theory, we obtain a model (on the same category) of the latter theory by Theorem 4.6. By construction, the empty type is inhabited in this model exactly if it is inhabited in the old model. \Box

Corollary 4.8. In Martin-Löf type theory with function extensionality and univalence for (n - 1)-types (and any further

n-benign type formers implemented by a known model of homotopy type theory), it is consistent to assume that all types *n*-truncated.

Proof. Apply Corollary 4.7 to a model of homotopy type theory such as simplicial sets or cubical sets that supports higher inductive families.

5 A Cubical Type Theory with UIP, Propositional Extensionality, and Homotopy Canonicity

In [14] the authors establish the homotopy canonicity property for a cubical type theory without judgmental equations for the box filling operations. Here we will follow that proof to prove homotopy canonicity for a cubical type theory with an axiomatic UIP principle and propositional extensionality given by a modified Glue-type. To keep this section brief, we closely follow their notation.

5.1 0-truncated Cubical Cwf

We take the definition of cubical cwf from [14] and adapt it by adding a new trunc operation and an extra argument to Glue-types. The definition is internal to the category of cubical sets of [11]. A minor difference to [14], following the previous section, we only require a sequential diagram of Ty_i presheaves, without topmost Ty, and we do not require the lifting map $Ty_i \rightarrow Ty_{i+1}$ to be mono.

Given $A : Ty_i(\Gamma)$, we define isProp(A) and isSet(A) in $Ty_i(\Gamma)$ using the Path-type former.

- **Glue types.** Given $A : \operatorname{Ty}_i(\Gamma), A_p : \operatorname{Elem}(\Gamma, \operatorname{isProp}(A)), \varphi : \mathbb{F}, T : [\varphi] \to \operatorname{Ty}_i(\Gamma), \text{ and } e : \operatorname{Elem}(\Gamma, \operatorname{Equiv}(T \operatorname{tt}, A)), we have the$ *glueing* $Glue(A, A_p, \varphi, T, e) in \operatorname{Ty}_i(\Gamma), equal to T tt on <math>\varphi$. We also have glue(a, t), unglue, and their equations as described in [14, Sec. 1.3].
- 0-truncation operation. Given A in Ty_i(Γ) we have trunc(A) in Elem(Γ, isSet(A)). No equations are required other than stability under substitution.

5.2 Standard Model

We now work in the category of cubical sets of [11]. It satisfies the assumptions listed at the top of [14, Section 2.2], so we have a hierarchy of universes of fibrant types U_i^{fib} , defined from a cumulative hierarchy of universes of presheaves U_i for $i \in \{0, 1, ..., \omega\}$. Using it as a model of cubical type theory with uniformly indexed higher inductive types, we replay the construction from Section 3 and obtain a family inV : $(U_i^{fib})^{\leq 0} \rightarrow U_{i+1}^{fib}$. Then we take

$$V_i \equiv \sum_{(A:(\bigcup_{i=1}^{\text{fib}}) \leq 0)} \text{inV}(A) : \bigcup_{i=1}^{\text{fib}}$$

with $\mathsf{El}_{\mathsf{V}_i}: \mathsf{V} \to (\mathsf{U}_i^{\mathsf{fib}})^{\leq 0}$.

Just as [14, Section 2.3] defines the standard model as an internal cwf from the universes U_i^{fib} , we define the standard model from the universes V_i :

 $^{^{\}overline{7}}$ This is the only place in our entire construction where judgmental β laws for higher inductive types are needed. One might well regard it as an artifact of our universe hierarchy setup.

- Con is the category with objects in U_ω and functions between them as morphism,
- the types of size *i* over $\Gamma : U_{\omega}$ are maps $\Gamma \to V_i$,
- the elements of $A : \Gamma \to V_i$ are $\Pi(\rho : \Gamma).\mathsf{El}_{V_i}(A\rho)$.

We will often omit the use of El_{V_i} to lighten the notational burden.

Type formers Π , Σ , N, Path and universes are given by including a code for them in the container (*S*, *Pos*) for V_{*i*}. The filling operation is derived from the one for \bigcup_{i}^{fib} . Glue-types are handled below.

5.2.1 A Code for Glue in V_i . One would think that inV might need an explicit constructor for Glue. However, the path constructor ua of inV suffices to derive one, given Glue for U^{fib} and fibrancy of inV.

Given $\Gamma : U_{\omega}, A : \Gamma \to V_i, A_p : \Pi(\rho : \Gamma).$ is Prop $(A \rho), \varphi : \mathbb{F}$, $T : [\varphi] \to \Gamma \to V_i$ and $e : [\varphi] \to$ Equiv(Ttt, A), we wish to define Glue $(A, A_p, \varphi, T, e) : \Gamma \to V_i$. We take

$$\mathsf{Glue}(A, A_p, \varphi, T, e) \rho \equiv_{\mathsf{def}} (G, w_G)$$

where $G = \text{Glue}(\text{El}_{V}(A \rho), \varphi, \text{El}_{V} \circ (T \rho), e \rho) : \bigcup_{i}^{\text{fib}}$, and given $w_{A} \equiv A \rho.2$ and $w_{T} \equiv \lambda o. T \circ \rho.2$, we obtain w_{Glue} by first transporting $w_{A} : \text{inV}(\text{El}_{V}(A \rho))$ to $w'_{G} : \text{inV}(G)$ by the canonical path between the two indices, and then composing under $[\varphi]$ with a path between w'_{G} and w_{T} built by ua. The latter is possible because, assuming $[\varphi]$, both w'_{G} and w_{T} are codes for $\text{El}_{V}(T \text{ tt } \rho)$, which is propositional by $A_{p} \rho$. Note that with this correction $\text{Glue}(A, A_{p}, \varphi, T, e) \equiv T$ tt when $[\varphi] \equiv \top$. One then checks that the code so defined commutes with the lifting maps $V_{i} \rightarrow V_{i+1}$.

5.3 Sconing Model

Given a cubical cwf \mathcal{M} (denoted by Con, Ty_i, Elem, ...), we want to define a new cubical cwf \mathcal{M}^* , (denoted by Con^{*}, Ty^{*}_i, Elem^{*}, ...) as the Arting glueing of \mathcal{M} along an internal global sections functor |-|. We assume \mathcal{M} size-compatible with the universes U_i as in [14, Sec. 3]. In [14], the functor |-| targets the standard model directly, given that Elem(1, \mathcal{A}) is a fibrant type. In our case, we have to include a code for it in inV_i, as in extending the container (*S*, *Pos*), as follows:

• given $A : Ty_i(1)$, a constructor $[A] : inV_i(Elem(1, A))$.

Note that both $Ty_i(1)$ and Elem(1, A) are 0-truncated, because the trunc operation implies isSet(Elem(1, A)) for any A in $Ty_j(1)$, and $Ty_i(1)$ itself is equivalent to $Elem(1, U_i)$. This makes sure that the extended container still preserves 0-truncatedness.

For the definition of natural numbers in \mathcal{M}^* , we will also need a code for the type family \mathbb{N}' : Elem $(1, N) \rightarrow U_0^{\text{fib}}$ defined in [14, Appendix B]:

- given n : Elem(1, N), a constructor N'(n) : inV₀(N'n)
- where \mathbb{N}' *n* can be shown to be 0-truncated by Corollary 3.14.

The functor |-| is then given on contexts, types, and elements of \mathcal{M} like so:

- $|\Gamma| \equiv_{\text{def}} \text{Hom}_{\mathcal{M}}(1, \Gamma),$
- $|A| \rho \equiv_{def} (Elem(1, A), \lceil A \rho \rceil),$
- $|a| \rho \equiv_{\text{def}} a\rho$.

We now define the sconing model \mathcal{M}^* , starting with the cwf components.

- A context (Γ, Γ') : Con^{*} consists of Γ : Con in \mathcal{M} and a family $\Gamma' : |\Gamma| \to \mathcal{U}_{\omega}$.
- A type (A, A') : $\mathsf{Ty}_i^*(\Gamma, \Gamma')$ consists of a type A : $\mathsf{Ty}_i(\Gamma)$ in \mathcal{M} and a family

$$A': \Pi(\rho:|\Gamma|)(\rho':\Gamma'\rho) \to |A|\rho \to \mathsf{V}_i$$

of proof-relevant predicates over it.

 An element (a, a') : Elem*((Γ, Γ'), (A, A')) consists of an element a : Elem(Γ, A) in M and

$$a': \Pi(\rho:|\Gamma|)(\rho':\Gamma'\rho) \to A(\rho,\rho',a\rho).$$

We observe that this definitions differs from the one given in Coquand et al. [14] only by the use of V_i in place of U_i^{fib} to define Ty_i^* . As such we will not repeat here the details about the rest of the cwf structure or the shared type formers and operations, and instead discuss only Glue^{*} and tr^{*}.

5.3.1 Glue-types.

Lemma 5.1. Let (A, A') in $Ty_i^*(\Gamma, \Gamma')$. The following statements are logically equivalent, naturally in (Γ, Γ') :

$$\mathsf{Elem}^*((\Gamma, \Gamma'), \mathsf{isProp}^*(A, A')) \tag{7}$$

$$Elem(\Gamma, isProp(A))$$

$$\times \Pi(\rho : |\Gamma|)(\rho' : \Gamma' \rho).isProp(\Sigma(a : |A| \rho).A'(\rho, \rho' a))$$

$$Elem(\Gamma, isProp(A))$$

$$\times \Pi(\rho : |\Gamma|)(\rho' : \Gamma' \rho)(a : |A| \rho).isProp(A'(\rho, \rho' a))$$
(9)

Proof. Given (7), we have A_p : Elem(Γ, isProp(A)) and a proof that one can fill lines in A' over lines produced by $|A_p|$; that is enough to fill lines in the Σ-type in (8). From there, we derive (9): since $|A| \rho$ is propositional, any path from a to a is constant. Going back to (7) requires only to contract a path in $|A| \rho$.

Let (A, A') in $\operatorname{Ty}_i^*(\Gamma, \Gamma')$, (A_p, A'_p) in $\operatorname{Elem}^*((\Gamma, \Gamma'), \operatorname{isProp}^*(A, A'))$, φ in \mathbb{F} , $\langle T, T' \rangle$ in $[\varphi] \to \operatorname{Ty}_i^*(\Gamma, \Gamma')$, and $\langle e, e' \rangle$ in $\operatorname{Elem}^*((\Gamma, \Gamma'), \operatorname{Equiv}^*((T \operatorname{tt}, T' \operatorname{tt}), (A, A')))$. We follow the recipe of [14, Sec. 3.2.6] and define

$$\mathsf{Glue}^*((A, A'), (A_p, A'_p), \varphi, \langle T, T' \rangle, \langle e, e' \rangle)$$

as $(Glue(A, A_p, \varphi, T, e), G')$ where $G' \rho \rho' (glue(a, t))$ is defined as the Glue-type in V_i between $A' \rho \rho' a$ and T' tt $\rho \rho' (t$ tt) along φ . In our case we also have to provide a proof of isProp $(A' \rho \rho' a)$, which we obtain from (A_p, A'_p) by applying Lemma 5.1, going from (7) to (9).

5.3.2 trunc-operation.

Lemma 5.2. Let (A, A') : $\mathsf{Ty}_i^*(\Gamma, \Gamma')$. The following statements are logically equivalent, naturally in (Γ, Γ') :

$$\mathsf{Elem}^*((\Gamma, \Gamma'), \mathsf{isSet}^*(A, A')) \tag{10}$$

 $Elem(\Gamma, isSet(A))$

$$\times \Pi(\rho : |\Gamma|)(\rho' : \Gamma' \rho).\text{isSet}(\Sigma(a : |A| \rho).A'(\rho, \rho' a))$$

$$\text{Elem}(\Gamma, \text{isSet}(A))$$

$$(11)$$

$$\times \Pi(\rho : |\Gamma|)(\rho' : \Gamma'\rho)(a : |A|\rho).\mathsf{isSet}(A'(\rho, \rho'a))$$
⁽¹²⁾

Proof. This follows the same strategy as Lemma 5.1, except this time filling and contracting squares rather than lines.

Let
$$(A, A')$$
 : Ty^{*}_i (Γ, Γ') . We define
trunc^{*} (A, A') : Elem^{*} $((\Gamma, \Gamma'), isSet*(A, A))$

by applying Lemma 5.2 in the direction from (10) to (12). to the pair of trunc(A) and trunc' where

trunc'
$$\rho \rho' a$$
 : isSet(El_{V_i}($A'(\rho, \rho' a)$))

is given by $\mathsf{El}_{V_i}(A'(\rho, \rho' a)) : (\mathsf{U}_i^{\text{fib}})^{\leq 0}$.

Given the above constructions, one mechanically verifies the necessary laws to obtain the following statement.

Theorem 5.3 (Sconing). Given any 0-truncated cubical cwf \mathcal{M} that is size-compatible in the sense of [14, Sec. 3], the sconing \mathcal{M}^* is a 0-truncated cubical cwf with operations defined as above. We further have a morphism $\mathcal{M}^* \to \mathcal{M}$ of 0-truncated cubical cwfs given by the first projection. \Box

We state homotopy canonicity with reference to the initial 0-truncated cubical cwf I, whose existence and sizecompatibility is justified as in [14, Sec. 4]. The proof of the theorem also follows the argument given in that section.

Theorem 5.4 (Homotopy canonicity). In the internal language of the cubical sets category of [11], given a closed natural n : Elem(1, N) in the initial model I, we have a numeral $k : \mathbb{N}$ with p : Elem(1, Path(N, $n, S^k(0))$).

6 Related Work and Conclusion

6.1 Related Work

In the realm of type theories with UIP and function extensionality, XTT [33] is a non-univalent variant of CTT that takes the extra step of making UIP hold judgmentally, in the spirit of observational type theory (OTT) [3]. As formulated XTT does not provide propositional extensionality and requires a typecase operation within the theory for (strict) canonicity. OTT does include propositional extensionality, but only for a universe of propositions closed under a specific set of type formers that made it possible to assume judgmental proof irrelevance for such propositions. We conjecture that by introducing UIP (or *n*-truncatedness) only as a path equality we will be able to refine our theory to one with strict canonicity without encountering similar limitations. Regarding strict propositions, i.e. where any two elements are stricly equal in the model, the semantics for a univalent universe of them within the cubical sets model is described in [12]. However the corresponding universe of strict sets is not a strict set itself. Such semantics are used in [18] to justify the addition of a primitive universe of strict propositions sProp.

6.2 Conclusion

We proved consistency for a theory with *n*-truncatedness and univalence for (n-1)-types. We also showed homotopy canonicity for cubical variant of such a theory. The main technical tool used was an *n*-truncated universe of *n*-types that is also univalent for (n - 1)-types. We would like to stress that such a universe can also be used directly in HoTT with indexed higher inductive types, for applications that do not mind the universe being limited to a fixed set of type formers.

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A Pushouts Along Monomorphisms

In this subsection, we prove some useful statements about pushouts along monomorphisms, in particular the fact that n-truncatedness of objects is preserved (Proposition A.11). We were unable to locate this statement in the higher topos literature. We expect that in homotopy type theory, many of these statements can also be obtained from [24] (pushouts being an example of a non-recursive higher inductive type), albeit under less minimalistic assumptions.

We use the language of higher categories (modelled for example by quasicategories [20, 27]). The statements of importance for the main body are about locally Cartesian closed higher categories. In homotopy type theory, this corresponds to the presence of identity types, dependent sums, and dependent products with function extensionality. All these statements and their proofs can also be read in homotopy type theory with these type formers, seen as an internal language for locally Cartesian closed higher categories (with pushout squares treated axiomatically as in [31]). It is in this form that they are used in the main body of the paper.

A map $V \to U$ in a higher category *C* is *univalent* if for any map $Y \to X$, the space of pullback squares from $Y \to X$ to $V \to U$ is (-1)-truncated, i.e. if the object $V \to U$ is (-1)truncated in the higher category C_{cart}^{\rightarrow} of arrows and Cartesian morphisms.⁸ We note the following for locally Cartesian closed *C*.

Given univalent V → U and an object Z, the pullback
 Z × V → Z × U is univalent in the slice over Z. This is a consequence of pullback pasting.

⁸We choose this definition as it makes sense in any higher category C.

The internal object of pullback squares (constructed using exponentials) from *Y* → *X* to univalent *V* → *U* is (-1)-truncated. To see this, one considers the space of pullback squares from *Z* × *Y* → *Z* × *X* to *V* → *U* for an arbitrary object *Z*.

Combining these observations, we may phrase univalence via the generic case: in the slice over U, the internal object of pullback squares from $V \rightarrow U$ to $U \times V \rightarrow U \times U$ (living over U via the first projection) is (-1)-truncated, or equivalently terminal.⁹ This is the univalence axiom for the "universe" U and the "universal family of elements" $V \rightarrow U$ in homotopy type theory: given x : U, the dependent sum of y : U and $V(x) \simeq V(y)$ is contractible, in turn equivalent to Definition 2.1. It also corresponds to the definition of univalent family given in [17]. The connection between univalent universes in the original sense of Voevodsky and univalent maps in our sense was probably first recognized by Joyal.

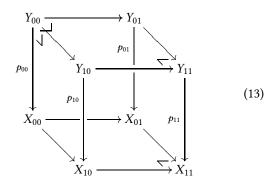
A *classifier* for a collection of maps $Y_i \to X_i$ for $i \in I$ is a univalent map $V \to U$ having $Y_i \to X_i$ as pullback for $i \in I$:



An object X is said to be classified if the map $X \to 1$ is classified. In homotopy type theory, a classifier for $(f_i)_{i \in I}$ (with external indexing) is simply a family V over U that satisfies the univalence axiom and restricts (up to equivalence) to the given families f_i for $i \in I$.

As discussed in [27], classifiers are related to *descent* for colimits in the sense of [30]. Here, we are interested only in the case of pushouts. Instead of assuming blanket classifiers for classes of "small" maps as in a higher topos, we will keep track of which collections of maps need to have classifiers.

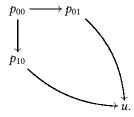
Proposition A.1 (Pushout descent from classifier). *Let C be a locally Cartesian closed higher category. Let*



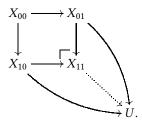
be a cube in C with horizontal faces pushouts and left and back faces pullbacks (as indicated). Assume that the maps p_{01} and p_{10} have a common classifier $V \rightarrow U$. Then the right and front faces are pullbacks and the map p_{11} is classified by $V \rightarrow U$.

Proof. This is a standard exercise, we give the proof for completeness.

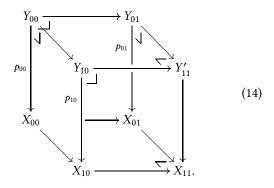
Let $u: V \to U$ be the given classifier. In C_{cart}^{\to} , by (-1)-truncatedness of u, we obtain a square



On codomains, we obtain an induced map



This allows us to see the bottom face of (13) as a square in the slice over *U*. Pulling back along *u*, we obtain a cube



⁹ The reverse implication proceeds as follows. Given $Y \to X$, let us show that the space of pullback squares from $Y \to X$ to $V \to U$ is (-1)truncated. We may suppose that $Y \to X$ is classified by $V \to U$ via a map $X \to U$. The desired space is isomorphic to the space of pullback squares over U from $Y \to X$ to $U \times V \to U \times U$. Over U, the map $Y \to X$ is the product of $V \to U$ with X, so this space is the hom space over U from Xto the internal object of pullback squares from $V \to U$ to $U \times V \to U \times U$. The target object of this hom space was assumed terminal.

Here, the top square is a pushout since pullback along u preserves pushouts, a consequence of local Cartesian closure. The cubes (13) and (14), seen from the top as squares in C^{\rightarrow} , are pushouts of the same span $p_{01} \leftarrow p_{00} \rightarrow p_{10}$. Pushouts are unique up to isomorphism, so the cubes are isomorphic. As the right and front faces are pullbacks in (14), so are they in (13). By construction, $Y'_{11} \rightarrow X_{11}$ is classified by $V \rightarrow U$, hence so is p_{11} .

Let us clarify the connection of Proposition A.1 to descent. A finitely complete higher category C has *descent* for a pushout square

if the slice functor C/- from $C^{\rm op}$ to higher categories sends it to a pullback square

$$C/X_{11} \longrightarrow C/X_{10}$$

$$\downarrow \qquad \qquad \downarrow$$

$$C/X_{01} \longrightarrow C/X_{00}$$

(morphisms are pullback functors). If this is the case, then:

- in any a cube (13) extending (15), the front and back faces are pullbacks (as in Proposition A.1),
- if *C* has pushouts of pullbacks of $X_{00} \rightarrow X_{10}$, the pushout (15) is stable under pullback.

If *C* has pushouts of pullbacks of $X_{00} \rightarrow X_{10}$, these conditions are equivalent to descent for (15).

Note that local Cartesian closure implies stability of all colimits, in particular pushouts, under pullback. In that case, Proposition A.1 shows that descent for a pushout follows from the existence of classifiers for finite collections of maps and the existence of certain pushouts. However, we eschew such a strong assumption on classifiers.

We will directly assume descent for pushouts for a track of results (Lemma A.2, Proposition A.4, Remark A.5, Lemma A.8, Proposition A.10, and Remark A.12) which cannot easily be expressed in terms of classifiers for some fixed maps or serve as inspiration for the setting using classifiers for a restricted choice of maps.

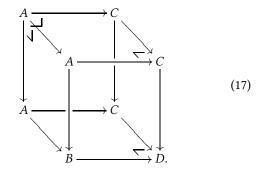
We will now prove a series of facts (some of them standard) about pushouts along monomorphisms, i.e. (-1)-truncated maps. These are sometimes also called embeddings.

Lemma A.2. In a finitely complete higher category with descent for pushouts, consider a pushout

If $A \to B$ mono, then so is $C \to B$ and the square is a pullback.

This is standard. We give its proof so that we can translate it to the setting with limited assumptions on classifiers.

Proof of Lemma A.2. Consider the cube



It is given by the functorial action on the map of arrows from $A \rightarrow B$ to $C \rightarrow D$ of the "connection operation" turning an arrow $X \rightarrow Y$ into a square



The map $X \rightarrow Y$ is mono exactly if this square is a pullback.

Let us inspect the faces of (17). The bottom face is the original pushout. The top face is a pushout and the back face is a pullback since opposite edges are invertible. The left face is a pullback since $A \rightarrow B$ is mono. By descent, it follows that the right and front faces are pullbacks. From the right face, we infer that $C \rightarrow D$ is mono. From the front face, we infer that (16) is a pullback.

Let us reprove this result starting from a classifier. Note the extended conclusion.

Lemma A.3. In a locally Cartesian closed higher category, let $A \rightarrow B$ be a monomorphism admitting a classifier $V \rightarrow U$ that also classifies the terminal object. Then for any pushout

$$\begin{array}{c}
A \longrightarrow C \\
\downarrow & \downarrow \\
B \longrightarrow D,
\end{array}$$
(18)

the map $C \rightarrow D$ is mono and classified by $V \rightarrow U$ and the square is a pullback.

Proof. We follow the proof of Lemma A.2. When it comes to using descent in the cube (17), we apply Proposition A.1. This uses that $A \rightarrow B$ and $C \rightarrow C$ (a pullback of $1 \rightarrow 1$) are classified by $V \rightarrow U$ and additionally gives that $C \rightarrow D$ is classified by $V \rightarrow U$.

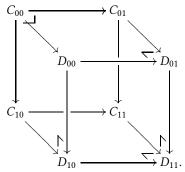
Lemma A.2 has the following interesting consequence.

Proposition A.4. In a finitely complete higher category with descent for pushouts, pushouts along monomorphisms preserve pullbacks (assuming the involved pushouts exist).

Proof. Recall that the forgetful functor from a coslice creates pullbacks. Consider a monomorphism $A \rightarrow B$ and a pullback square



under A. We consider its pushout along $A \rightarrow B$, forming a cube



Our goal is to show that its front face is a pullback. By Lemma A.2, follows from a double invocation of Lemma A.6. the maps from back to front are mono. By pushout pasting, the faces from back to front are pushouts. By Lemma A.2, the left face is a pullback. Hence, the front face is a pullback by descent.

Remark A.5. Let *C* be a finitely complete higher category with descent for pushouts. Given a monomorphism $A \rightarrow B$ having pushouts, Proposition A.4 shows that the pushout functor $C \setminus A \to C \setminus D$ is almost left exact, with only the terminal object not generally being preserved. We can pass to slices to fix this: given a pushout square as in (18), the induced functor from the higher category of factorizations of $A \rightarrow C$ to the higher category of factorizations of $B \rightarrow D$ is left exact.

For the following statement, note that the forgetful functor from a slice higher category creates truncation levels of maps. That is, the truncation level of a map does not change if we view regard the map as living over different objects.

Lemma A.6. In a finitely complete higher category, let

be a pushout stable under pullback. A map over D is n-truncated exactly if its pullbacks over B and C are n-truncated.

Proof. We induct on *n*. In the base case n = -2, this is functoriality of pushouts that exist. The case $n \ge -1$ reduces to

the case for n - 1 since diagonals of maps are preserved by pullbacks.

The following lemma is a simple recognition criterion for truncation levels of pushouts.

Lemma A.7. In a finitely complete higher category, let

$$\begin{array}{c}
A \longrightarrow C \\
\downarrow \qquad & \downarrow \\
B \longrightarrow D
\end{array}$$
(20)

be a pushout stable under pullback. Let $n \ge -1$. Then D is n-truncated exactly if the maps

(i) $B \times_D B \to B \times B$, (*ii*) $B \times_D C \to B \times C$, (iii) $C \times_D C \to C \times C$ are (n-1)-truncated.

Proof. D is *n*-truncated exactly if $D \rightarrow D \times D$ is (n-1)truncated. By pullback pasting, one sees that the listed maps are the pullbacks of the diagonal $D \rightarrow D \times D$ along the map $L \times R \rightarrow D \times D$ for all combinations of L and R being B or C (the map (ii) appears twice). Thus, they are (n-1)-truncated if $D \rightarrow D \times D$ is (n - 1)-truncated. The reverse implication

The *join* $X \star Y$ of objects X and Y, if it exists, is their pushout product, seen as maps $X \rightarrow 1$ and $Y \rightarrow 1$, i.e. the pushout

Lemma A.8. In a finitely complete higher category with descent for pushouts, assume that the pushout (21) exists and is stable under pullback. If X and Y are (-1)-truncated, then so is their join $X \star Y$.

Proof. By assumption, the left and top maps in (21) are mono. Applying Lemma A.2 twice, the square is a pullback and the right and bottom maps are mono. We check that $X \star Y$ is (-1)-truncated by instantiating Lemma A.7 to the pushout (21).

- Since $X \to X \star Y$ is mono, (i) is the diagonal $X \to X \star Y$ $X \times X$. This is an equivalence since X is (-1)-truncated.
- Since (21) is a pullback, (ii) is an equivalence.
- The case for (iii) is analogous to the one for (i), using that $Y \rightarrow X \star Y$ is mono and Y is (-1)-truncated. \Box

We have an analogous statement in the setting with classifiers, proved using Lemma A.3 instead of Lemma A.2.

Lemma A.9 (Propositional Join). In a locally Cartesian closed higher category, assume objects X and Y have classifiers also classifying the terminal object. If X and Y are (-1)-truncated, then so is their join $X \star Y$ if it exists.

For the application of Lemma A.9 in the main body of the paper, we note that the assumptions on classifiers are satisfied in homotopy type theory as soon as we have univalence for propositions.

We finally come to the main result of this section. We first state it using descent. In this case, it has an easier proof.

Proposition A.10. Consider a finitely complete higher category with descent for pushouts. Let $A \rightarrow B$ be a monomorphism having pushouts that are stable under pullback. Consider a pushout

If B and C are n-truncated for $n \ge 0$, then so is D. This also holds for n = -1 if we have a map $B \times C \rightarrow A$.

Proof. We proceed by induction on n. Of note, the inductive step will apply the claim to a slice, so the ambient category changes within the induction. (In homotopy type theory, this simply corresponds to a use of the induction hypothesis in an extended context.)

In the base case n = -1, note that A is (-1)-truncated since $A \rightarrow B$ is mono and B is (-1)-truncated. Since we have maps back and forth between A and $B \times C$ and both objects are (-1)-truncated, these maps are invertible. It follows that the span in (22) is a product span, and the claim reduces to Lemma A.8.

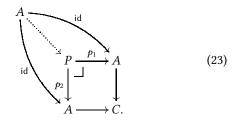
From now on, assume $n \ge 0$. By Lemma A.2, the map $C \rightarrow D$ is mono and (22) is a pullback. We check that *D* is *n*-truncated by instantiating Lemma A.7 to the pushout (22).

- Since $C \to D$ is mono, the map (iii) is the diagonal $C \to C \times C$. This is (n 1)-truncated since C is *n*-truncated.
- Since (22) is a pullback, the map (ii) is A → B×C. This factors as

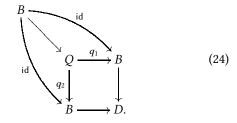
$$A \longrightarrow A \times C \longrightarrow B \times C.$$

The first factor is a pullback of the diagonal $C \rightarrow C \times C$, which is (n - 1)-truncated since C *n*-truncated. The second factor is a pullback of $A \rightarrow B$, which is (n - 1)truncated since $n \ge 0$ and $A \rightarrow B$ is mono.

The remaining case is the map (i), i.e. showing that $B \times_D B \rightarrow B \times B$ is (n - 1)-truncated. Consider the diagram producing the diagonal of *A* over *C*:



We see this as a square in the coslice under *A*. We take its pushout along $A \rightarrow B$:



We will independently prove the following:

(i) the inner square in (24) is a pullback,

(ii) the map $\langle q_1, q_2 \rangle \colon Q \to B \times B$ is (n-1)-truncated.

Together, this implies the goal.

Let us prove (i). By pushout pasting, the squares

are pushouts. By Lemma A.2, the map $P \rightarrow Q$ is mono. Again by Lemma A.2, the squares (25) are pullbacks. Applying descent in the cube connecting the inner squares in (23) and (24), we see that the inner square in (24) is a pullback.

Let us prove (ii). Focus on the pushout square

$$\begin{array}{c} A \longrightarrow P \\ \downarrow \qquad \qquad \downarrow \\ B \longrightarrow Q. \end{array}$$

$$(26)$$

It lives in the slice over $B \times B$ via $\langle q_1, q_2 \rangle$. The induced map $P \rightarrow B \times B$ rewrites as

$$P \xrightarrow{\langle p_1, p_2 \rangle} A \times A \longrightarrow B \times B.$$

As in the proof of Lemma A.7, the first factor is a pullback of the diagonal $C \rightarrow C \times C$, hence (n - 1)-truncated. The second factor is mono, hence (n - 1)-truncated since $n \ge 0$. It follows that P is (n - 1)-truncated over $B \times B$. Note that B is (n - 1)-truncated over $B \times B$ by assumption.

We apply the induction hypothesis to the pushout (26) in the slice over $B \times B$. Note that the assumptions on descent and the mono $A \rightarrow B$ all descend to the slice. We have just shown that the bottom left and top right objects of (26) are (n - 1)-truncated over $B \times B$. For the case n - 1 = -1, we note that we have maps

$$B \times_{B \times B} P \longrightarrow B \times_{B \times B} (A \times A) \xleftarrow{\simeq} A$$

over $B \times B$ where the second map inverts as $A \to B$ is mono. This finishes the proof of (ii).

We now state the above result in terms of classifiers. Since we only assume classifiers for (n - 1)-truncated maps, the previous proof requires modification. **Proposition A.11** (*n*-truncated Pushout Of Mono). In a locally Cartesian closed higher category, let $m: A \rightarrow B$ be a monomorphism having pushouts. Consider a pushout

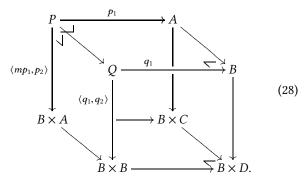
Let $n \ge 0$. Assume a classifier for any finite collection of (n-1)truncated maps. If B and C are n-truncated, then so is D. This also holds for n = -1 if we have a map $B \times C \rightarrow A$ and a classifier for any finite collection of monomorphisms.

Proof. The proof proceeds as for Proposition A.10, using Lemma A.9 instead of Lemma A.8 in the base case n = -1.

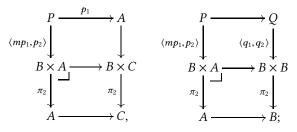
In the situation for $n \ge 0$, we use Lemma A.3 instead of Lemma A.2 to deduce that $C \rightarrow D$ is mono and (27) is a pullback.

A divergence occurs when checking the subgoals (i) and (ii). The previous descent argument for (i) no longer works because the relevant maps in the cube considered are not (n - 1)-truncated. Instead, we are forced to prove (i) in a way that will depend on (ii). The proof of (ii) proceeds as before, noting that the assumptions on classifiers descend to the recursive case.

Let us now prove (i). To start, we use Lemma A.3 instead of Lemma A.2 to derive the assertions about the squares (25). Consider the cube

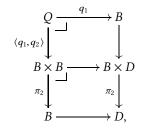


The bottom face is the pullback of (27) along $B \rightarrow 1$, hence a pushout by local Cartesian closure. The top face is one of the pushouts (25). To see that the back and left faces are pullbacks, pullback paste in the diagrams



here, the left composite square is a pullback by construction (23) and the right composite square is one of the pullbacks (25).

We already know that $A \rightarrow B \times C$ is (n-1)-truncated. The map $Q \rightarrow B \times B$ is (n-1)-truncated by (ii). By assumption, they have a common classifier. We are thus in the position to apply Proposition A.1 to the cube (28) and deduce that its front face is a pullback. Pasting the pullbacks



we get (i).

Remark A.12. Following the idea of the proof of Proposition A.11, one may strengthen also the statement of Proposition A.10 by restricting descent to (n - 1)-descent (or (-1)-descent for n = -1). Here, *n*-descent in a finitely complete higher category *C* refers to the descent-like notion obtained by instead considering the functor from C^{op} to higher categories that sends *Z* to the full higher subcategory of C/Z on *n*-truncated objects.

Remark A.13. It is possible to state the assumption on classifiers in Proposition A.11 in a form closer to current systems of homotopy type theory. This involves introducing a notion of universe, i.e. fixing a (not necessarily univalent) map $V \rightarrow U$ whose classified maps are closed under composition and formation of diagonals and whose classified objects in any slice are closed under pushouts. One then requires A, B, C to be classified by $V \rightarrow U$ and that the collection of those maps classified by $V \rightarrow U$ that are (n - 1)-truncated (or (-1)-truncated for n = -1) also admits a (univalent) classifier in our sense.

Note that we are speaking here about classification in the sense of higher categories, not strict classification by some universe in a model of type theory. This is the essential difference to the topic of the main body of the article where this statement is used.

For the application of Proposition A.11 in the main body of the paper, we note that the assumptions on classifiers are satisfied in homotopy type theory as soon as we have univalence for propositions.

B Partially Propositional Indexed W-types

In this section, we develop the theory of partially propositional indexed W-types and characterize their equality. As a warm up, we recall indexed containers, the associated notion of indexed-type, and the encode-decode method used to characterize their equality. The results developed here will be used in Section 3 in the definition of an *n*-truncated universe of *n*-types that is univalent for (n - 1)-types.

We work informally in the language of homotopy type theory. In particular, we have access to function extensionality. We will be explicit about universes when they are needed and what kind of univalence we require of them.

No result in this section depends on judgmental β -equality for higher inductive types, even for point constructors. The β -law as an internal equality will suffice. This applies to pushouts, the higher inductive families that will constitute partially propositional W-types, and even ordinary indexed W-types.

B.1 Indexed Containers

Given a type *I*, an *I*-indexed container *C* is a pair C = (S, Pos) of type families as follows:

- given i : I, we have a type S(i),
- given i, j : I, and s : S(i), we have a type Pos(s, j).¹⁰

Its *extension* is the endofunctor on families over I that sends a family X to the family Ext(X) given by

$$\mathsf{Ext}(X)(i) = \sum_{(s:S(i))} \prod_{(j:I)} \mathsf{Pos}(i, s, j) \to X(j)$$

B.2 Indexed W-types

Fix an *I*-indexed container *C* as above. As in [6, 21] (the former applying only to the non-indexed case), one has notions of *algebras*, *algebra morphisms*, *algebra fibrations*, and *algebra fibration sections* for *C*. Here and in the following, we omit the prefix "homotopy" for all relevant notions, this being the default meaning for us.

Definition B.1 (Indexed W-type). A *W*-type for the indexed container *C* is an initial *C*-algebra, meaning the type of algebra morphisms to any algebra is contractible,

We denote a given, substitutionally stable choice of such an object by (W_C , sup), although denoting the whole algebra by W_C . One may characterize W_C via elimination: any algebra fibration over W_C has an algebra section. This is known as *induction*. Note that the β -law for the eliminator holds only up to identity type.

In a framework with inductive families (with or without judgmental β -law), one may implement W_C as an inductive family with constructor $\sup(s, t) : W_C(i)$ for i : I, s : S(i), and $t : \prod_{(j:I)} Pos(s, j) \to W_C(j)$; when applying t, we leave its first argument j implicit. To keep the presentation readable, we will informally use W_C as if it was given as such an inductive family and use pattern-matching-style notation for induction; we leave the reduction to the eliminator (including the definition of the algebra fibration corresponding to each use of induction) to the reader.

B.3 Encode-Decode Method

The encode-decode method [25] is a general method for characterizing equality in a (higher) inductive type (or family) T. In our view, it decomposes into the following three steps (here for just a single type T).

- 1. Define a binary relation Eq_T of *equality codes* on *T*. This uses double induction on *T* (with ultimate target a universe) and makes use of univalence if there are any path constructors.
- 2. Define an "encoding" function

$$T(x_0, x_1) \xrightarrow{\operatorname{encode}_{x_0, x_1}} \mathsf{Eq}_T(x_0, x_1).$$
⁽²⁹⁾

This uses induction over the given equality and single induction on *T*.

3. Prove $encode^{-1}(c)$ for each code $c : Eq_T(x_0, x_1)$. This is a pair (p, q) where $p : T(x_0, x_1)$ and q : encode(p) = c. This uses double induction on *T* as in step 1.

Step 3 makes (29) into a retraction. Summing over $x_1 : T$, the source of the retraction becomes contractible. But any retraction with contractible source is an equivalence. By a standard lemma about fiberwise equivalences [34, Thm 4.7.7], it follows that the original map (29) is an equivalence.

B.4 Equality in Indexed W-types

We now apply the encode-decode method from Subsection B.3 to characterize equality in W_C . Although we did not find this in the literature, we believe it to be folklore. We have a choice between two viable options:

- characterize equality in each fiber $W_C(i)$ for i : I,
- characterize dependent equality in W_C over a given equality in I.

We go for the second one.

For step 1, we define a type $Eq(p_i, x_0, x_1)$ of *equality codes* between $x_0 : W_c(i_0)$ and $x_1 : W_C(i_1)$ over $p_i : I(i_0, i_1)$. We proceed by double induction on x_0 and x_1 into a large enough universe. For $x_0 \equiv sup(s_0, t_0)$ and $x_1 \equiv sup(s_1, t_1)$, we take $Eq(p_i, x_0, x_1)$ equal to the type of pairs (p_s, c_t) where:

- p_s is a dependent equality in *S* over p_i between s_0 and s_1 ,
- c_t is a dependent function, sending j : I¹¹, m₀ : Pos(s₀, j), m₁ : Pos(s₁, j), and a dependent equality p_m over p_i and p_s between m₀ and m₁ to

$$c_t(p_m)$$
: Eq(refl_j, $t_0(m_0), t_1(m_1))$.

Steps 2 and 3 will establish the following statement.

Proposition B.2. Given $p : I(i_0, i_1)$ with $x_0 : W_C(i_0)$ and $x_1 : W_C(i_1)$, we have

$$(x_0 =_{W_C(p)} x_1) \simeq Eq(p, x_0, x_1).$$

¹⁰This is a polynomial functor $I \leftarrow E \rightarrow B \rightarrow I$ with Reedy fibrant specifying data. The latter means that $B \rightarrow I$ and $E \rightarrow B \times I$ are fibrations.

¹¹Instead of j : I, we could also quantify over $j_0, j_1 : I$ with $p_j : I(j_0, j_1)$. Perhaps this would be more consistent.

Proof. For step 2, we first define $encode'(x) : Eq(refl_i, x, x)$ for i : I and $x : W_C(i)$. We proceed by induction on x, letting $x \equiv sup(s, t)$. We let encode'(x) be $(refl_s, p_t)$, transported along the β -equality for Eq. Here, p_t is defined by equality induction from $p_t(j, refl_m) \equiv_{def} encode'(t(m))$. From encode', we obtain

$$x_0 =_{W_C(p_i)} x_1 \xrightarrow{\operatorname{encode}_{p_i, x_0, x_1}} \operatorname{Eq}(p_i, x_0, x_1)$$

by equality induction first on p_i and then the equality between x_0 and x_1 in the same fiber.

For step 3, we prove $\operatorname{encode}_{p_i, x_0, x_1}^{-1}(c)$ for $c : \operatorname{Eq}(p_i, x_0, x_1)$. Inducting on p_i , we may suppose $i \equiv_{\operatorname{def}} i_0 \equiv i_1$ and $p_i \equiv$ refl. We induct on x_0 and x_1 , letting $x_0 \equiv \sup(s_0, t_0)$ and $x_1 \equiv \sup(s_1, t_1)$. Using the β -equality for Eq, we can reduce to the case where c is the transport of a pair (p_s, c_t) as in step 1. Inducting on p_s , we may suppose $s \equiv_{\operatorname{def}} s_0 \equiv s_1$ and $p_s \equiv$ refl.

Given *j* : *I*, note that the canonical map

$$Pos(s, j) \rightarrow \sum_{(m_0, m_1: Pos(s, j))} m_0 = m_1$$

is an equivalence. Using function extensionality and equality induction, we may thus suppose that c_t is obtained using equality induction from

$$c'_t(m) \equiv_{\text{def}} c_t(\text{refl}_m) : \text{Eq}(\text{refl}_i, t_0(m), t_1(m))$$

for j : I and m : Pos(s, j) in the same fashion that p_t is defined in the definition of encode'. By induction hypothesis, we have $encode_{refl_j, t_0(m), t_1(m)}^{-1}(c'_t(m))$. Using function extensionality and equality induction, we may thus suppose that

$$c'_t \equiv \lambda j. \lambda m. \text{ encode}_{\text{refl}_i, t_0(m), t_1(m)}(q_m(m))$$

for some $q_m(m)$: $t_0(m) =_{W_C(j)} t_1(m)$ depending on j : Iand m : Pos(s, j). Again using function extensionality and equality induction, we may suppose that $t \equiv_{def} t_0 \equiv t_1$ and $q_m \equiv \lambda j. \lambda m. \operatorname{refl}_{t(m)}$.

Now we have $c'_t \equiv \lambda j$. λm . encode'(t(m)), and thus

$$c \equiv (\operatorname{refl}_{s}, c_{t})$$

$$\equiv \operatorname{encode}'(\sup(s, t))$$

$$\equiv \operatorname{encode}_{\operatorname{refl}_{i}, \sup(s, t), \sup(s, t)}(\operatorname{refl}).$$

This shows encode $p_{i,x_0,x_1}^{-1}(c)$.

B.5 Partially Propositional Indexed W-types

Fix an *I*-indexed container *C*. In addition, fix a family P of propositions over *I*.

We wish to mix the inductive construction of the indexed W-type W_C with the propositional truncation, applied only over indices i : I with P(i). Note that indexed W-types are not merely fiberwise W-types. Thus, it would be wrong to take W_C and simply conditionally truncate each fiber, Rather, we want to have the truncation interleaved within the inductive definition.

A type *A* is *Q*-propositional if *A* is propositional assuming Q, i.e., if $Q \rightarrow isProp(A)$. A family *X* over *I* is *P*-propositional if X(i) is P(i)-propositional for i : I. We use analogous terminology with being contractible.

A *C*-algebra is *P*-propositional if the underlying family is *P*-propositional.

Definition B.3 (Partially propositional indexed W-type). A *partially propositional W-type* for the propositional family *P* and the indexed container *C* is an initial *P*-propositional *C*-algebra.

We denote a given, substitutionally stable choice of such an object by W_C^P . We also call this the *P*-propositional indexed W-type generated by by *C*. As before, one may characterize $W_{C,P}$ via elimination (i.e. induction): any *P*-propositional algebra fibration over W_C^P has an algebra section. Here, an algebra fibration over a *P*-propositional algebra is *P*-propositional if the total space algebra is *P*-propositional.

Equivalently, W_C^P is the higher inductive family [21, 26] with point constructor sup as for W_C and path constructor

$$\operatorname{tr}(c, x, y) : W_C^P(i)(x, y)$$

for i : I with c : P(i) and $x, y : W_C^P(i)$. As before, we do not assume that β -equality is judgmental. Again, to keep the presentation readable, we will informally use induction on W_C^P using pattern-matching-style notation, leaving the compilation to elimination with respect to *P*-propositional algebra fibrations to the reader. The reasoning principle is the same as for W_C , only that we must show the target's fiber over i : I is propositional if P(i).

Higher inductive families have been justified in the simplicial set model [26] and various cubical sets models [10, 13] of homotopy type theory. Thus, these models support partially propositional W-types (even with judgmental β -reduction).

B.6 Equality in Partially Propositional Indexed W-types

We adapt the encode-decode method from Subsection B.3 to characterize equality in W_C^P . Large parts of the construction will follow the technical development in Subsection B.4, so the focus will lie on the new aspect of *P*-propositionality.

In step 1, we need to define codes for equality. This relies on the following lemma.

Lemma B.4. Let \mathcal{U} be a universe univalent for contractible types. Let Q be proposition. Then the subtype of \mathcal{U} of Q-contractible types is Q-contractible.

Proof. If Q holds, Q-contractibility means contractibility. So the goal becomes: if Q holds, the subtype of \mathcal{U} of contractible types is contractible. But the latter is always contractible, using univalence.

Given $p_i : I(i_0, i_1)$ with $x_0 : W_c^P(i_0)$ and $x_1 : W_C^P(i_1)$, we define simultaneously:

- a type $Eq(p_i, x_0, x_1)$ of equality codes,
- a witness that $Eq(p_i, x_0, x_1)$ is $P(i_0)$ -contractible.¹²

We proceed by double elimination on x_0 and x_1 with target over i : I the type of P(i)-contractible types in a large enough universe \mathcal{U} univalent for contractible types. This is a *P*-propositional algebra by Lemma B.4. For $x_0 \equiv \sup(s_0, t_0)$ and $x_1 \equiv \sup(s_1, t_1)$, we take the join

$$\mathsf{Eq}(p_i, x_0, x_1) = P(i_0) \star \mathsf{Eq}'(p_i, (s_0, t_0), (s_1, t_1)).$$
(30)

Here, $Eq'(p_i, (s_0, t_0), (s_1, t_1))$ codes structural equality of the top-level constructor applications. It is defined as the type of pairs (p_s, c_t) as in the construction of Eq in Subsection B.4, with c_t ultimately valued in Eq. The join (30) may also be understood as a $P(i_0)$ -partial contractible truncation. In particular, it is contractible if $P(i_0)$.

Proposition B.5. Given $p : I(i_0, i_1)$ with $x_0 : W_C^P(i_0)$ and $x_1 : W_C^P(i_1)$, we have

$$(x_0 =_{W_{\mathcal{O}}^P(p)} x_1) \simeq Eq(p, x_0, x_1).$$

Proof. It remains to adapt steps 2 and 3 of the encode-decode method.

For step 2, we first define $encode'(x) : Eq(refl_i, x, x)$ for i : I and $x : W_C(i)$. Note that the goal is contractible if P(i). We may thus induct on x. For $x \equiv sup(s, t)$, we define encode'(x) = inr(...) where the omitted expression is as in the constructor case for encode' in Proposition B.2. From encode', obtain

$$x_0 =_{W_C^P(p_i)} x_1 \xrightarrow{\operatorname{encode}_{p_i, x_0, x_1}} \mathsf{Eq}(x_0, x_1)$$

as in Proposition B.2.

For step 3, we prove $\operatorname{encode}_{p_i, x_0, x_1}^{-1}(c)$ for $c : \operatorname{Eq}(p_i, x_0, x_1)$. Inducting on p_i , we may suppose $i \equiv_{\operatorname{def}} i_0 \equiv i_1$ and $p_i \equiv$ refl. Note that the goal becomes contractible if P(i): source and target of $\operatorname{encode}_{\operatorname{refl}_i, x_0, x_1}$ become contractible. Thus, we may induct on x_0 and x_1 , letting $x_0 \equiv \sup(s_0, t_0)$ and $x_1 \equiv \sup(s_1, t_1)$.

Using the β -equality for Eq, we can reduce to the case where *c* is the transport of an element of the join (30). We eliminate over this element of the join. In the case for inl or glue, we have P(i), so are done as the goal is contractible. In the case for inr, we have that *c* is the transport of $inr(p_s, c_t)$ where (p_s, c_t) is as in Proposition B.2. The rest of the proof follows that proof.

Remark B.6. An evident generaliation of partially propositional indexed W-types is *partially n-truncated indexed W-types* for internal or external $n \ge -2$. We expect that the encode-decode method can also be adapted to characterize equality in partially *n*-truncated indexed W-types. Instead

of using the join as in Proposition B.5, one defines $Eq(p_i, x_0, x_1)$ as the pushout

$$P(i_{0}) \times Eq'(p_{i}, (s_{0}, t_{0}), (s_{1}, t_{1}))$$

$$P(i_{0}) \times ||Eq'(p_{i}, (s_{0}, t_{0}), (s_{1}, t_{1}))||_{n-1}$$

$$Eq'(p_{i}, (s_{0}, t_{0}), (s_{1}, t_{1}))$$

$$Eq(p_{i}, x_{0}, x_{1}).$$

This is motivated by the equality in an *n*-truncation being the (n - 1)-truncation of the original equality. For n = -1, this definition reduces to Proposition B.5. For n = -2, the definition still makes sense if we take the (-3)-truncation to mean (-2)-truncation.

Note that there is an induced map

$$\mathsf{Eq}(p_i, x_0, x_1) \to \|\mathsf{Eq}'(p_i, (s_0, t_0), (s_1, t_1))\|_{n-1}.$$
 (31)

This is the pushout product of $P(i_0) \rightarrow 1$ with the map

$$\mathsf{Eq}'(p_i, (s_0, t_0), (s_1, t_1)) \to \|\mathsf{Eq}'(p_i, (s_0, t_0), (s_1, t_1))\|_{n-1}$$
(32)

Thus, the fiber of (31) over an element e is given by the join of $P(i_0)$ and the fiber of (32) over e. Thus, the join still plays a role, even though it is hidden in a different slice.

¹²Note that $P(i_0) \leftrightarrow P(i_1)$ given $p_i : I(i_0, i_1)$.