# A Complete Proof System for 1-Free Regular Expressions Modulo Bisimilarity 

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#### Abstract

Robin Milner (1984) gave a sound proof system for bisimilarity of regular expressions interpreted as processes: Basic Process Algebra with unary Kleene star iteration, deadlock 0 , successful termination 1, and a fixed-point rule. He asked whether this system is complete. Despite intensive research over the last 35 years, the problem is still open.

This paper gives a partial positive answer to Milner's problem. We prove that the adaptation of Milner's system over the subclass of regular expressions that arises by dropping the constant 1 , and by changing to binary Kleene star iteration is complete. The crucial tool we use is a graph structure property that guarantees expressibility of a process graph by a regular expression, and is preserved by going over from a process graph to its bisimulation collapse.


Keywords regular expressions, process algebra, bisimilarity, process graphs, complete proof system

## 1 Introduction

Regular expressions, introduced by Kleene [17], are widely studied in formal language theory, notably for string searching [29]. They are constructed from constants 0 (no strings), 1 (the empty string), and a (a single letter) from some alphabet; binary operators + and $\cdot$ (union and concatenation); and the unary Kleene star * (zero or more iterations).

Their interpretations are Kleene algebras with as prime example the algebra of regular events, the language semantics of regular expressions, which is closely linked with deterministic finite state automata. Aanderaa [1] and Salomaa [24] gave complete axiomatizations for the language semantics of regular expressions, with a non-algebraic fixed-point rule that has a non-empty-word property as side condition. Krob [20] gave an infinitary, and then Kozen [18] a finitary algebraic axiomatization involving equational implications.

Regular expressions also received significant attention in the process algebra community [5], where they are interpreted modulo the bisimulation process semantics [22]. Robin Milner [21] was the first to study regular expressions in this setting, where he called them star expressions. Here the interpretation of 0 is deadlock, 1 is (successful) termination, $a$ is an atomic action, and + and are alternative

[^0]and sequential composition of two processes, respectively. Milner adapted Salomaa's axiomatization to obtain a sound proof system for this setting, and posed the (still open) question whether this axiomatization is complete, meaning that if the process graphs of two star expressions are bisimilar, then they can be proven equal.

Milner's axiomatization contains a fixed-point rule, which is inevitable because due to the presence of 0 the underlying equational theory is not finitely based [25, 26]. Bergstra, Bethke, and Ponse [4] studied star expressions without 0 and 1 , replaced the unary by the binary Kleene star ${ }^{\circledast}$, which represents an iteration of the first argument, possibly eventually followed by the execution of the second argument. They obtained an axiomatization by basically omitting the axioms for 0 and 1 as well as the fixed-point rule from Milner's axiomatization, and adding Troeger's axiom [30]. This purely equational axiomatization was proven complete in [9, 11]. A sound and complete axiomatization for star expressions without unary Kleene star, but with 0 and 1 and a unary perpetual loop operator ${ }^{*} 0$ (equivalently, unary star is restricted to terms $e^{*} \cdot 0$ ), was given in $[8,10]$.

In contrast to the formal languages setting, not all finitestate process graphs can be expressed by a star expression modulo bisimilarity. Milner posed a second question in [21], namely, to characterize which finite-state process graphs can be expressed. This was shown to be decidable in [3] by defining and using 'well-behaved' specifications.

In this paper we prove completeness of Milner's axiomatization (tailored to the adapted setting) for star expressions with 0 , but without 1 and with the binary Kleene star.

While earlier completeness proofs focus on manipulation of terms, we follow Milner's footsteps and focus on their process graphs. A key idea is to determine loops in graphs associated to star expressions. By a loop we mean a subgraph generated by a set of entry transitions from a vertex $v$ in which (1) there is an infinite path from $v$, (2) each infinite path eventually returns to $v$, and (3) termination is not permitted. A graph is said to satisfy LLEE (Layered Loop Existence and Elimination) if repeatedly eliminating the entry transitions of a loop, and performing garbage collection, leads to a graph without infinite paths. LLEE offers a generalization (and more elegant definition) of the notion of a well-behaved specification.

Our completeness proof roughly works as follows (for more details see Sect. 4). Let $e_{1}$ and $e_{2}$ be star expressions that have bisimilar graphs process graph interpretations $g_{1}$ and $g_{2}$. We show that $g_{1}$ and $g_{2}$ satisfy LLEE. We moreover prove that LLEE is preserved under bisimulation collapse. And we construct for each graph that satisfies LLEE a star expression that corresponds to this graph, modulo bisimilarity. In particular such a star expression $f$ can be constructed for the bisimulation collapse of $g_{1}$ and $g_{2}$. We show that both $e_{1}$ and $e_{2}$ can be proven equal to $f$, by a pull-back over the functional bisimulations from the bisimulation collapse back to $g_{1}$ and $g_{2}$. This yields the desired completeness result.

In our proof, the minimization of terms (and thereby of the associated process graphs) in the left-hand side of a binary Kleene star modulo bisimilarity is partly inspired by [8, 10]. Interestingly, we will be able to use as running example the process graph interpretation of the star expression that at the end of [10] is mentioned as problematic for a completeness proof. Our crucial use of witnesses for the graph property LLEE borrows from the representation of cyclic $\lambda$-terms [15] as structure-constrained term graphs, as used for defining and implementing maximal sharing in the $\lambda$-calculus with letrec [16] (see also [13]).

The completeness result for star expressions with 0 but without 1 and with the binary Kleene star settles a natural question. We are also hopeful that the property LLEE provides a strong conceptual tool for approaching Milner's long-standing open question regarding the class of all star expressions. The presence of 1 -transitions in graphs presents new challenges, such as that LLEE is not always preserved under bisimulation collapse. In order to be able to still work with this concept, we will need workarounds.

This is a report version of the article [14] in the proceedings of the conference LICS 2020. It was compiled from the submission version, containing a technical appendix.

Please see the appendix for details of proofs that have been omitted or that are only sketched.

## 2 Preliminaries

In this section we define star expressions, their process semantics as 'charts', the proof system BBP for bisimilarity of their chart interpretations, and provable solutions of charts.

Definition 2.1. Given a set $A$ of actions, the set $\operatorname{StExp}(A)$ of star expressions over $A$ is generated by the grammar:
$e::=0|a|\left(e_{1}+e_{2}\right)\left|\left(e_{1} \cdot e_{2}\right)\right|\left(e_{1}{ }^{\circledast} e_{2}\right) \quad$ (with $\left.a \in A\right)$.
0 represents deadlock (i.e., does not perform any action), $a$ an atomic action, + alternative and $\cdot$ sequential composition, and ${ }^{\circledast}$ the binary Kleene star. Note that 1 (for empty steps) is missing from the syntax. $\sum_{i=1}^{k} e_{i}$ is defined recursively as 0 if $k=0, e_{1}$ if $k=1$, and $\left(\sum_{i=1}^{k-1} e_{i}\right)+e_{k}$ if $k>1$.

The star height $|e|_{\circledast}$ of a star expression $e \in \operatorname{StExp}(A)$ denotes the maximum number of nestings of Kleene stars in $e$ :
it is defined by $|0|_{\circledast}:=|a|_{\circledast}:=0,|f+g|_{\circledast}:=|f \cdot g|_{\circledast}:=$ $\max \left\{|f|_{\circledast},|g|_{\circledast}\right\}$, and $\left|f^{\circledast} g\right|_{\circledast}:=\max \left\{|f|_{\circledast}+1,|g|_{\circledast}\right\}$.

Definition 2.2. By a (finite sink-termination) chart $C$ we understand a 5 -tuple $\left\langle V, \sqrt{ }, v_{s}, A, T\right\rangle$ where $V$ is a finite set of vertices, $\sqrt{ }$ is, in case $\sqrt{ } \in V$, a special vertex with no outgoing transitions (a sink) that indicates termination (in case $\sqrt{ } \notin V$, the chart does not admit termination), $v_{\mathrm{s}} \in V \backslash\{\sqrt{ }\}$ is the start vertex, $A$ is a set actions, and $T \subseteq V \times A \times V$ the set of transitions. Since $A$ can be reconstructed from $T$, we will frequently keep $A$ implicit, denote a chart as a 4 -tuple $\left\langle V, \sqrt{ }, v_{\mathrm{s}}, T\right\rangle$. A chart is start-vertex connected if every vertex is reachable by a path from the start vertex. This property can be achieved by removing unreachable vertices ('garbage collection'). We will assume charts to be start-vertex connected.

In a chart $C$, let $v \in V$ and $U \subseteq T$ be a set of transitions from $v$. By the $\langle v, U\rangle$-generated subchart of $C$ we mean the chart $C_{0}=\left\langle V_{0}, \sqrt{ }, v, A, T_{0}\right\rangle$ with start vertex $v$ where $V_{0}$ is the set of vertices and $T_{0}$ the set of transitions that are on paths in $C$ from $v$ that first take a transition in $U$, and then, until $v$ is reached again, continue with other transitions of $C$. We use the standard notation $v \xrightarrow{a} v^{\prime}$ in lieu of $\left\langle w, a, w^{\prime}\right\rangle \in T$.

Definition 2.3. Let $C_{i}=\left\langle V_{i}, \sqrt{ }, v_{\mathrm{s}, i}, T_{i}\right\rangle$ for $i \in\{1,2\}$ be two charts. A bisimulation between $C_{1}$ and $C_{2}$ is a relation $B \subseteq V_{1} \times V_{2}$ that satisfies the following conditions:
(start) $v_{\mathrm{s}, 1} B v_{\mathrm{s}, 2}$ (it relates the start vertices), and for all $v_{1}, v_{2} \in V$ with $v_{1} B v_{2}$ :
(forth) for every transition $v_{1} \xrightarrow{a} v_{1}^{\prime}$ in $C_{1}$ there is a transition $v_{2} \xrightarrow{a} v_{2}^{\prime}$ in $C_{2}$ with $v_{1}^{\prime} B v_{2}^{\prime}$,
(back) for every transition $v_{2} \xrightarrow{a} v_{2}^{\prime}$ in $C_{2}$ there is a transition $v_{1} \xrightarrow{a} v_{1}^{\prime}$ in $C_{1}$ with $v_{1}^{\prime} B v_{2}^{\prime}$,
(termination) $v_{1}=\sqrt{ }$ if and only if $v_{2}=\sqrt{ }$.
If there is a bisimulation between $C_{1}$ and $C_{2}$, then we write $C_{1} \leftrightarrows C_{2}$ and say that $C_{1}$ and $C_{2}$ are bisimilar. If a bisimulation is the graph of a function, we say that it is a functional bisimulation. We write $\mathcal{C}_{1} \Longrightarrow C_{2}$ if there is a functional bisimulation between $C_{1}$ and $C_{2}$.

Definition 2.4. For every star expression $e \in \operatorname{StExp}(A)$ the chart interpretation $C(e)=\langle V(e), \sqrt{ }, e, A, T(e)\rangle$ of $e$ is the chart with start vertex $e$ that is specified by iteration via the following transition rules, which form a transition system specification (TSS), with $e, e_{1}, e_{2}, e_{1}^{\prime} \in \operatorname{StExp}(A), a \in A$ :

$$
\begin{gathered}
\underset{a \xrightarrow{a} \sqrt{ }}{ } \frac{e_{i} \xrightarrow{a} \xi}{e_{1}+e_{2} \xrightarrow{a} \xi}(i=1,2) \\
\frac{e_{1} \xrightarrow{a} e_{1}^{\prime}}{e_{1} \cdot e_{2} \xrightarrow{a} e_{1}^{\prime} \cdot e_{2}} \frac{e_{1} \xrightarrow{a} \sqrt{ }}{e_{1} \cdot e_{2} \xrightarrow{a} e_{2}} \\
\frac{e_{1} \xrightarrow{a} e_{1}^{\prime}}{e_{1} \circledast e_{2} \xrightarrow{a} e_{1}^{\prime} \cdot\left(e_{1} \circledast e_{2}\right)} \frac{e_{1} \xrightarrow{a} \sqrt{ }}{e_{1} \circledast e_{2} \xrightarrow{a} e_{1} \circledast e_{2}} \xrightarrow[{e_{1} \circledast e_{2} \xrightarrow{a}} \xi]{e^{3}}
\end{gathered}
$$

with $\xi \in \operatorname{StExp}(A)_{\sqrt{ }}:=\operatorname{StExp}(A) \cup\{\sqrt{ }\}$, where $\sqrt{ }$ indicates sink termination. If $e \xrightarrow{a} \xi$ can be proved, $\xi$ is called an $a$-derivative, or just derivative, of $e$. The set $V(e) \subseteq \operatorname{StExp}(A)_{\sqrt{ }}$ consists of the iterated derivatives of $e$. To see that $C(e)$ is finite, Antimirov's result [2], that a regular expression has only finitely many iterated derivatives, can be adapted.

We say that a star expression $e \in \operatorname{StExp}(A)$ is normed if there is a path of transitions from $e$ to $\sqrt{ }$ in $C(e)$.


Example 2.5. By the rules in Def. 2.4, $e_{0}:=a \cdot e_{0}^{\prime}$ with $e_{0}^{\prime}:=$ $(c \cdot a+a \cdot(b+b \cdot a)))^{\circledast} 0$ has the $\operatorname{chart} C\left(e_{0}\right)$ as above, with $v_{0}:=e_{0}, v_{1}:=e_{0}^{\prime}$ and $v_{2}:=(b+b \cdot a) \cdot e_{0}^{\prime}$. This chart is the bisimulation collapse of the charts $C\left(e_{1}\right)$ and $C\left(e_{2}\right)$ of star expressions $e_{1}:=\left(a \cdot\left((a \cdot(b+b \cdot a))^{\left.\circledast_{c}\right)}\right)^{\circledast_{0}}\right.$, and $e_{2}:=a \cdot\left(\left(c \cdot a+a \cdot\left(b \cdot a \cdot\left((c \cdot a)^{\circledast} a\right)\right)^{\circledast} b\right)^{\circledast} 0\right)$. Bisimulations between $\mathcal{C}\left(e_{1}\right)$ and $C\left(e_{0}\right)$, and between $C\left(e_{0}\right)$ and $C\left(e_{1}\right)$ are indicated by the broken lines. The chart $\mathcal{C}\left(e_{0}\right)$ was considered problematic in [10].

Example 2.6. The left chart below does not admit termination. The right chart is a double-exit graph with the sink termination vertex $\sqrt{ }$ at the bottom.


These charts are not bisimilar to chart interpretations of star expressions. For the left chart this was shown by Milner [21], and for the right chart by Bosscher [6].

Definition 2.7. The proof system BBP or the class of star expressions has the axioms (B1)-(B6), (BKS1), (BKS2), the inference rules of equational logic, and the rule RSP ${ }^{*}$ :

$$
\begin{align*}
x+y & =y+x  \tag{B1}\\
(x+y)+z & =x+(y+z)  \tag{B2}\\
x+x & =x  \tag{B3}\\
(x+y) \cdot z & =x \cdot z+y \cdot z \tag{B4}
\end{align*}
$$

$$
\begin{align*}
(x \cdot y) \cdot z & =x \cdot(y \cdot z)  \tag{B5}\\
x+0 & =x  \tag{B6}\\
0 \cdot x & =0  \tag{B7}\\
x \cdot\left(x^{\circledast} y\right)+y & =x^{\circledast} y  \tag{BKS1}\\
\left(x^{\circledast} y\right) \cdot z & =x^{\circledast}(y \cdot z)  \tag{BKS2}\\
x & =(y \cdot x)+z \\
\hline x & =y^{\circledast} z
\end{align*}
$$

By $e_{1}={ }_{\text {BBP }} e_{2}$ we denote that $e_{1}=e_{2}$ is derivable in BBP.
BBP is a finite 'implicational' proof system [28], because unlike in Salomaa's and Milner's systems for regular expressions with 1 the fixed-point rule does not require any side-condition to ensure 'guardedness'.

Definition 2.8. For a chart $C=\left\langle V, \sqrt{ }, v_{\mathrm{s}}, A, T\right\rangle$, a provable solution of $C$ is a function $s: V \backslash\{\sqrt{ }\} \rightarrow \operatorname{StExp}(A)$ such that:

$$
s(v)=\mathrm{BBP}\left(\sum_{i=1}^{m} a_{i}\right)+\left(\sum_{j=1}^{n} b_{j} \cdot s\left(w_{j}\right)\right) \quad(\text { for all } v \in V \backslash\{\sqrt{ }\})
$$

holds, given that the union of $\left\{v \xrightarrow{a_{i}} \sqrt{ } \mid i=1, \ldots, m\right\}$ and $\left\{v \xrightarrow{b_{j}} w_{j} \mid j=1, \ldots, n, w_{j} \neq \sqrt{ }\right\}$ is the set of transitions from $v$ in $C$. We call $s\left(v_{\mathrm{s}}\right)$ the principal value of $s$.
Proposition 2.9 (uses BBP-axioms (B1)-(B7), (BKS1)). For every $e \in \operatorname{StExp}(A)$, the identity function $\operatorname{id}_{V(e)}: V(e) \rightarrow$ $V(e) \subseteq \operatorname{StExp}(A), e^{\prime} \mapsto e^{\prime}$, is a provable solution of the chart interpretation $C(e)$ of $e$.

Proof (Idea). Each $e$ in $\operatorname{StExp}(A)$ is the BBP-provable sum of expressions $a$ and $a \cdot e^{\prime}$ over all $a \in A$ for $a$-derivatives $\sqrt{ }$ and $e^{\prime}$, respectively, of $e$. This 'fundamental theorem ${ }^{1}$ of differential calculus for star expressions' implies, quite directly, that $\mathrm{id}_{V(e)}$ is a provable solution of $C(e)$.

## 3 Layered loop existence and elimination

As preparation for the definition of the central concept of 'LLEE-witness', we start with an informal explanation of the structural chart property 'LEE'. It is a necessary condition for a chart to be the chart interpretation of a star expression. LEE is defined by a dynamic elimination procedure that analyses the structure of the graph by peeling off 'loop subcharts'. Such subcharts capture, within the chart interpretation of a star expression $e$, the behaviour of the iteration of $f_{1}$ within innermost subterms $f_{1} \circledast f_{2}$ in $e$. (A weaker form of 'loop' by Milner [21], which describes the behavior of general iteration subterms, is not sufficient for our aims.)

Definition 3.1. A chart $\mathcal{L}=\left\langle V, \sqrt{ }, v_{\mathrm{s}}, T\right\rangle$ is a loop chart if: (L1) There is an infinite path from the start vertex $v_{\mathrm{s}}$.

[^1](L2) Every infinite path from $v_{\mathrm{s}}$ returns to $v_{\mathrm{s}}$ after a positive number of transitions (and so visits $v_{\mathrm{s}}$ infinitely often).
(L3) $V$ does not contain the vertex $\sqrt{ }$.
In such a loop chart we call the transitions from $v_{\mathrm{s}}$ loop-entry transitions, and all other transitions loop-body transitions.

Let $C$ be a chart. A loop chart $\mathcal{L}$ is called a loop subchart of $C$ if $\mathcal{L}$ is the $\langle v, U\rangle$-generated subchart of $C$ for some vertex $v$ of $C$, and a set $U$ of transitions of $C$ that depart from $v$ (so the transitions in $U$ are the loop-entry transitions of $\mathcal{L})$.

Note that the two charts in Ex. 2.6 are not loop charts: the left one violates (L2), and the right one violates (L3). Moreover, none of these charts contains a loop subchart. While the chart $C\left(e_{0}\right)$ in Ex. 2.5 is not a loop chart either, as it violates (L2), we will see that it has loop subcharts.

Let $\mathcal{L}$ be a loop subchart of a chart $\mathcal{C}$. Then the result of eliminating $\mathcal{L}$ from $C$ arises by removing all loop-entry transitions of $\mathcal{L}$ from $C$, and then removing all vertices and transitions that get unreachable. We say that a chart $C$ has the loop existence and elimination property (LEE) if the process, started on $C$, of repeated eliminations of loop subcharts results in a chart that does not have an infinite path.

For the charts in Ex. 2.6 the procedure stops immediately, as they do not contain loop subcharts. Since both of them have infinite paths, it follows that they do not satisfy LEE.

We consider three runs of the elimination procedure for the chart $\mathcal{C}\left(e_{0}\right)$ in Ex. 2.5. The loop-entry transitions of loop subcharts that are removed in each step are marked in bold.


Each run witnesses that $C$ satisfies LEE. Note that loop elimination does not yield a unique result. ${ }^{2}$ Runs can be recorded by attaching, in the original chart, to transitions that get removed in the elimination procedure as marking label the sequence number of the appertaining elimination step. For

[^2]the three runs of loop elimination above we get the following marking labeled versions of $C$, respectively:


Since all three runs were successful (as they yield charts without infinite paths), these recordings (marking-labeled charts) can be viewed as 'LEE-witnesses'. We now will define a concept of a 'layered LEE-witness' (LLEE-witness), i.e., a LEE-witness with the added constraint that in the formulated run of the loop elimination procedure it never happens that a loop-entry transition is removed from within the body of a previously removed loop subchart. This refined concept has simpler properties, and it will fit our purpose.

Before introducing 'LLEE-witnesses', we first define chart labelings that mark transitions in a chart as '(loop-)entry' and as '(loop-)body' transitions, but without safeguarding that these markings refer to actual loops.

Definition 3.2. Let $C=\left\langle V, v_{s}, \sqrt{ }, A, T\right\rangle$ be a chart. An en-try/body-labeling $\hat{C}=\left\langle V, v_{\mathrm{s}}, \sqrt{ }, A \times \mathbb{N}, \widehat{T}\right\rangle$ of $C$ is a chart that arises from $C$ by adding, for each transition $\tau=\left\langle v_{1}, a, v_{2}\right\rangle \in$ $T$, to the action label $a$ of $\tau$ a marking label $\alpha \in \mathbb{N}$, yielding $\hat{\tau}=\left\langle v_{1},\langle a, \alpha\rangle, v_{2}\right\rangle \in \widehat{T}$. In such an entry/body-labeling we call transitions with marking label 0 body transitions, and transitions with marking labels in $\mathbb{N}^{+}$entry transitions.

Let $\hat{C}$ be an entry/body-labeling of $C$, and let $v$ and $w$ be vertices of $C$ and $\hat{C}$. We denote by $v \rightarrow_{\text {bo }} w$ that there is a body transition $v \xrightarrow{\langle a, 0\rangle} w$ in $\hat{C}$ for some $a \in A$, and by $v \rightarrow[\alpha] w$, for $\alpha \in \mathbb{N}^{+}$that there is an entry transition $v \xrightarrow{\langle a, \alpha\rangle} w$ in $\hat{C}$ for some $a \in A$. We will use $\alpha, \beta, \gamma, \ldots$ for marking labels in $\mathbb{N}^{+}$of entry transitions. By the set $E(\hat{C})$ of entry transition identifiers we denote the set of pairs $\langle v, \alpha\rangle \in V \times \mathbb{N}^{+}$such that an entry transition $\rightarrow{ }_{[\alpha]}$ departs from $v$ in $\hat{C}$. For $\langle v, \alpha\rangle \in E(\hat{\mathcal{C}})$, we define by $C_{\hat{C}}(v, \alpha)$ the subchart of $C$ with start vertex $v_{s}$ that consists of the vertices and transitions which occur on paths in $C$ as follows: they start with a $\rightarrow_{[\alpha]}$ entry transition from $v$, continue with body transitions only, and halt immediately if $v$ is revisited.

Definition 3.3. A LLEE-witness $\hat{C}$ of a chart $C$ is an entry/ body-labeling of $C$ that satisfies the following properties:
(W1) There is no infinite path of $\rightarrow_{b o}$ transitions from $v_{\mathrm{s}}$.
(W2) For all $\langle v, \alpha\rangle \in E(\hat{C})$, (a) $C_{\hat{C}}(v, \alpha)$ is a loop chart, and (b) (layeredness) from no vertex $w \neq v$ of $C_{\hat{C}}(v, \alpha)$ there departs in $\hat{\mathcal{C}}$ an entry transition $\rightarrow[\beta]$ with $\beta \geqslant \alpha$.

The stipulation in (W2)(a) justifies to call entry transitions in a LLEE-witness a loop-entry transition. For a loop-entry transition $\rightarrow{ }_{[\beta]}$ with $\beta \in \mathbb{N}^{+}$, we call $\beta$ its loop level.

A chart is a LLEE-chart if it has a LLEE-witness.
Example 3.4. The three labelings of the chart $\mathcal{C}\left(e_{0}\right)$ in Ex. 2.5 that arose as recordings of runs of the loop elimination procedure can be viewed as entry/body-labelings of that chart. There, and below, we dropped the body labels of transitions, and instead only indicated the entry labels in boldface together with their levels. By checking conditions (W1) and (W2),(a)-(b), it is easy to verify that these entry/body-labelings are LLEE-witnesses. In fact it is not difficult to establish that every LLEE-witness of $C\left(e_{0}\right)$ in Ex. 2.5 is of either of the following two forms, with marking labels $\alpha, \beta, \gamma, \delta, \epsilon \in \mathbb{N}^{+}$:


We now argue that LLEE-witnesses guarantee the property LEE. Let $\hat{C}$ be a LLEE-witness of a chart $C$. Repeatedly pick an entry transition identifier $\langle v, \alpha\rangle$ with $\alpha \in \mathbb{N}^{+}$minimal, remove the loop subchart that is generated by loop-entry transitions of level $\alpha$ from $v$ (it is indeed a loop by (W2)(a), and minimality of $\alpha$ and (W2)(b) ensure the absence of departing loop-entry transitions of lower level), and perform garbage collection. Eventually the part of $C$ that is reachable by body transitions from the start vertex is obtained. This subchart does not have an infinite path due to (W1). Therefore $C$ indeed satisfies LEE, as witnessed by $\hat{C}$.

The property LEE and the concept of LLEE-witness are closely linked with the process semantics of star expressions. In fact, we now define a labeling of the TSS in Def. 2.4 that permits to define, for every star expression $e$, an entry/bodylabeling of the chart interpretation $C(e)$ of $e$, which can then be recognized as a LLEE-witness of $C(e)$.

We refine the TSS rules in Def. 2.4 as follows: A body label is added to transitions that cannot return to the star expression in their left-hand side. The rule for transitions into the iteration part $e_{1}$ of an iteration $e_{1}{ }^{\circledast} e_{2}$ is split into the cases where $e_{1}$ is normed or not. Only in the normed case can $e_{1}{ }^{\circledast} e_{2}$ return to itself, and then a loop-entry transition with the star height $\left|e_{1}\right|_{\circledast}$ of $e_{1}$ as its level is created.
Definition 3.5. For every $e \in \operatorname{StExp}(A)$, we define the en-try/body-labeling $\widehat{C(e)}$ of the chart interpretation $C(e)$ of $e$ in analogy with $C(e)$ by using the following transition rules that refine the rules in Def. 2.4 by adding marking labels:

$$
\frac{e_{i} \xrightarrow[\rightarrow]{a} \xi}{a \xrightarrow[\rightarrow]{a} \text { bo } \sqrt{ }} \quad \frac{e_{1}+e_{2} \xrightarrow{a} \text { bo } \xi}{e_{1}} i \in\{1,2\}
$$



Figure 1. LLEE-witness entry/body-labelings as defined by Def. 3.5 for the chart interpretations of $e_{0}, e_{1}$, and $e_{2}$ in Ex. 2.5.

$$
\begin{aligned}
& \frac{e_{1} \xrightarrow{a} l e_{1}^{\prime}}{e_{1} \cdot e_{2} \xrightarrow{a} l e_{1}^{\prime} \cdot e_{2}} \quad \frac{e_{1} \xrightarrow{a} \mathrm{bo} \sqrt{ }}{e_{1} \cdot e_{2} \xrightarrow{a} \mathrm{bo} e_{2}} \\
& \frac{e_{1} \xrightarrow{a} l e_{1}^{\prime}}{e_{1} \circledast e_{2} \xrightarrow{a}\left[\left|e_{1}\right| \circledast+1\right] \quad e_{1}^{\prime} \cdot\left(e_{1} \circledast e_{2}\right)} \text { if } e_{1} \text { is normed } \\
& \frac{e_{1} \xrightarrow{a} l e_{1}^{\prime}}{e_{1} \circledast e_{2} \xrightarrow{a} \text { bo } e_{1}^{\prime} \cdot\left(e_{1}^{*} e_{2}\right)} \text { if } e_{1} \text { is not normed } \\
& \frac{e_{1} \xrightarrow{a} \text { bo } \sqrt{ }}{e_{1} \circledast_{e_{2}} \xrightarrow{a}\left[\left|e_{1}\right| \circledast+1\right]} e_{1}{ }^{\circledast} e_{2} \quad \frac{e_{2} \xrightarrow{a} l \dot{a} \xi}{e_{1}{ }^{\circledast} e_{2} \xrightarrow{a} \text { bo } \xi}
\end{aligned}
$$

for $l \in\{$ bo $\} \cup\left\{[\alpha] \mid \alpha \in \mathbb{N}^{+}\right\}$, where we employed notation defined in Def. 2.4 for writing marking labels as subscripts.

Example 3.6. In Fig. 1 we depict the entry/body-labelings, as defined in Def. 3.2, for star expressions $e_{1}, e_{0}$, and $e_{2}$ in Ex. 2.5. It is easy to verify that these labelings are LLEE-witnesses of the charts $C\left(e_{0}\right), C\left(e_{1}\right)$, and $C\left(e_{2}\right)$ in Ex. 2.5, resp..
Proposition 3.7. For every $e \in \operatorname{StExp}(A)$, the entry/body-labeling $\widehat{C(e)}$ of $C(e)$ is a LLEE-witness of $C(e)$.

For a binary relation $R$, let $R^{+}$and $R^{*}$ be its transitive and transitive-reflexive closures. $u \rightarrow_{l} v$ denotes that there is a transition $u \xrightarrow{a} l v$ for an $a \in A$, and in proofs (but not pictures) $u \rightarrow v$ denotes that $u \rightarrow_{l} v$ for some label $l$. By $u \underset{t(w)}{\longrightarrow} v$ we denote that $u \rightarrow_{l} v$ and $v \neq w$ (this transition avoids target $w$ ). Likewise, $u \underset{ \pm(w)}{\longrightarrow} v$ denotes that $u \underset{ \pm(w)}{ } v$ for some label $l$. By $\operatorname{scc}(u)$ we denote the strongly connected component (scc) to which $u$ belongs.
Definition 3.8. Let $\hat{C}$ be a LLEE-witness of chart $\mathcal{C}$. If there is a path $v \underset{\ddagger(v)}{\longrightarrow}[\alpha] \cdot \xrightarrow[ \pm(v)]{ }{ }^{*}$ o $w$, then we write $v^{\alpha} \frown w$. (Note that $v^{\alpha} \frown w$ holds if and only if $w$ is a vertex $\neq v$ of the loop chart $C_{\hat{C}}(v, \alpha)$ that is generated by the $\rightarrow[\alpha]$ entry transitions at $v$ in $C$.) We write $v \frown w$ and say that $v$ descends in a loop to $w$ if $v^{\alpha} \frown w$ for some $\alpha \in \mathbb{N}^{+}$.

We write $w G v$ (or $v \bigcirc w$ ), and say that $w$ loops back to $v$, if $v \frown w \rightarrow_{\text {bo }}^{+} v$. The loops-back-to relation $G$ totally orders its successors (see Lem. 3.9, (vi)). Therefore we define the 'direct successor relation' ${ }_{d} G$ of $Q$ as follows: We write $w_{d} G v\left(\right.$ or $\left.v_{d} \bigcirc w\right)$, and say that $w$ directly loops back to $v$, if $w G v$ and for all $u$ with $w G$ either $u=v$ or $v G u$.

Lemma 3.9. The relations $\rightarrow{ }_{\mathrm{bo}}, \frown, G,{ }_{d} \mathrm{G}$ as defined by a LLEE-witness $\hat{\mathcal{C}}$ on a chart $\mathcal{C}$ satisfy the following properties:
(i) There are no infinite $\rightarrow_{\mathrm{bo}}$ paths (so no $\rightarrow_{\mathrm{bo}}$ cycles).
(ii) If $\operatorname{scc}(u)=\operatorname{scc}(v)$, then $u ゝ^{*} v$ implies $v G^{*} u$.
(iii) If $v \frown w$ and $\neg(w)$, then $w$ is not normed.
(iv) $\operatorname{scc}(u)=\operatorname{scc}(v)$ if and only if $u G^{*} w$ and $v G^{*} w$ for some vertex $w$.
(v) $Q^{*}$ is a partial order with the least-upper-bound property: if a nonempty set of vertices has an upper bound with respect to $G^{*}$, then it has a least upper bound.
(vi) $G$ is a total order on $G$-successor vertices: if $w G v_{1}$ and $w G v_{2}$, then $v_{1} G v_{2}$ or $v_{1}=v_{2}$ or $v_{2} G v_{1}$.
(vii) If $v_{1}{ }_{d} G u$ and $v_{2}{ }_{d} G u$ for distinct $v_{1}, v_{2}$, then there is no vertex $w$ such that both $w G^{*} v_{1}$ and $w G^{*} v_{2}$.

## 4 The completeness proof, anticipated

After having introduced LLEE-charts as our crucial auxiliary concept, we now sketch the completeness proof. In doing so we need to anticipate four results that will be developed in the next two sections: (C) The bisimulation collapse of a LLEE-chart is again a LLEE-chart. (E) From every LLEE-chart a provable solution can be extracted. (S) All provable solutions of LLEE-charts are provably equal. (P) All provable solutions can be pulled back from the target to the source chart of a functional bisimulation.

Then completeness of BBP can be argued as follows. Given two bisimilar star expressions $e_{1}$ and $e_{2}$, obtain their chart interpretations $\mathcal{C}\left(e_{1}\right)$ and $\mathcal{C}\left(e_{2}\right)$, which are LLEE-charts due to Prop. 3.7. By Prop. 2.9, $e_{1}$ and $e_{2}$ are principal values of provable solutions of $\mathcal{C}\left(e_{1}\right)$ and $\mathcal{C}\left(e_{2}\right)$. These charts have the same bisimulation collapse $C$. By (C, Thm. 6.9), $C$ is again a LLEE-chart. Use (E, Prop. 5.5) to build a provable solution $s$ of $C$; let its principal value be $e$. Apply ( $\mathbf{P}$, Prop. 5.1) to transfer $s$ backwards over the functional bisimulations to obtain provable solutions $s_{1}$ and $s_{2}$ of $C\left(e_{1}\right)$ and $C\left(e_{2}\right)$, respectively. By construction, $s_{1}$ and $s_{2}$ have the same principal value $e$ as $s$. Finally, by using (S, Prop. 5.8), $e_{1}$ and $e_{2}$ are both provably equal to $e$. Hence, $e_{1}=\operatorname{\text {ввр}} e=$ ввр $e_{2}$.

In his completeness proof for regular expressions in formal language theory, Salomaa [24] argued 'upwards' from two equivalent regular expressions to a larger regular expression that can be homomorphically collapsed onto both of them. In contrast, our proof approach forces us 'downwards' to the bisimulation collapse, because in the opposite direction the property of being a LLEE-chart may be lost.

Example 4.1. The picture below highlights why we cannot adopt Salomaa's proof strategy of linking two languageequivalent regular expressions via the product of the DFAs they represent. The bisimilar LLEE-charts $C_{1}$ and $C_{2}$ are interpretations of $(a \cdot(a+b)+b)^{{ }^{*}} 0$ and $(b \cdot(a+b)+a)^{{ }^{*}} 0$, respectively (the indicated labelings $\hat{\mathcal{C}}_{1}$ and $\hat{C}_{2}$ are LLEE-witnesses). But their product $C_{12}$ is a not a LLEE-chart; it is of the form of one the not expressible charts from Ex. 2.6. Yet their common bisimulation collapse $C_{0}$, the chart interpretation of $(a+b)^{\circledast} 0$, is a LLEE-chart with LLEE-witness $\hat{C}_{0}$.


In view of $C_{1} \leftrightarrows C_{12} \rightrightarrows C_{2}$ this also shows that LLEE-charts are not closed under converse functional bisimilarity $\leftrightarrows$.

## 5 Extraction of star expressions from, and transferral between, LLEE-charts

In this section we develop the results (E), (S), and (P) as mentioned in Sect. 4. We start with the statement (P).

Proposition 5.1 (requires BBP-axioms (B1), (B2), (B3)). Let $\phi: V_{1} \rightarrow V_{2}$ be a functional bisimulation between charts $C_{1}$ and $C_{2}$. If $s_{2}: V_{2} \backslash\{\sqrt{ }\} \rightarrow \operatorname{StExp}(A)$ is a provable solution of $C_{2}$, then $s_{2} \circ \phi: V_{1} \backslash\{\sqrt{ }\} \rightarrow \operatorname{StExp}(A)$ is a provable solution of $C_{1}$ with the same principal value as $s_{2}$.

Proof (Idea). The bisimulation clauses make it possible to demonstrate the condition for $s_{2} \circ \phi$ to be a provable solution of $C_{1}$ at $w$ by using the condition for the provable solution $s_{2}$ of $C_{2}$ at $\phi(w)$, together with the axioms (B1), (B2), (B3).

We now turn to proving results (E) and (S) from Sect. 4. We show that from every chart $C$ with LLEE-witness $\hat{C}$ a provable solution $s_{\hat{C}}$ of $C$ can be extracted. Intuitively, the extraction process follows a run of the loop-elimination procedure on $C$, guided by the LLEE-witness $\hat{C}$. All loop subcharts that are generated by the loop-entry transitions from a vertex $v$ are removed in a row. ${ }^{3}$ Extraction synthesizes a star expression $e_{1}$ whose behavior captures the eliminated loop subcharts of $v$ and their previously eliminated inner loop subcharts, and that will later be part of an iteration

[^3]expression $e_{1}{ }^{\circledast} e_{2}$ in the solution value at $v$. This idea motivates an inside-out extraction process that works with partial solutions, and eventually builds up a provable solution of $C$.

In particular, we inductively define 'relative extracted solutions' $t_{\hat{C}}(w, v)$ for vertices $v$ and $w$ where $w$ is in a loop subchart $C_{\hat{C}}(v, \alpha)$ at $v$, for some $\alpha \in \mathbb{N}^{+}$, that is, $v^{\alpha} \frown w$. Hereby $t_{\hat{C}}(w, v)$ captures the part of the behavior in $C$ from $w$ until $v$ is reached. Then we define the from $\hat{C}$ 'extracted solution' $s_{\hat{C}}(v)$ at $v$ by using the relative solutions $t_{\hat{C}}\left(w_{j}, v\right)$ for all targets $w_{j}$ of loop-entry transitions from $v$ to define the iteration part $e_{1}$ of the extracted solution $s_{\hat{C}}(v)=e_{1}{ }^{\circledast} e_{2}$ at $v$. We start with a preparation.

Let $\hat{C}$ be a LLEE-witness, and let $v$ be a vertex of $\hat{C}$. By the entry step level $|v|_{\text {en }}$ of $v$ we mean the maximum loop level of a loop-entry transition in $\hat{C}$ that departs from $v$, or 0 if no loop-entry transition departs from $v$. By the body step norm $\|v\|_{\text {bo }}$ of $v$ we mean the maximal length of a body transition path in $C$ from $v$ (well-defined by Lem. 3.9, (i)).

Lemma 5.2. For all vertices $v, w$ in a chart $C$ with LLEE-witness $\hat{C}$ it holds (for the concepts as defined with respect to $\hat{C}$ ):
(i) $v \rightarrow_{\mathrm{bo}} w \Rightarrow\|v\|_{\mathrm{bo}}>\|w\|_{\mathrm{bo}}$,
(ii) $v \frown w \Rightarrow|v|_{\mathrm{en}}>|w|_{\mathrm{en}}$.

Definition 5.3. Let $\hat{C}$ be a LLEE-witness of a chart $C$. Then the relative extraction function of $\hat{C}$ is defined inductively as:

$$
\begin{aligned}
& t_{\hat{\mathcal{C}}}:\{\langle w, v\rangle \mid v, w \in V \backslash\{\sqrt{ }\}, v \frown w\} \rightarrow \operatorname{StExp}(A), \\
& t_{\hat{\mathcal{C}}}(w, v):=\left(\left(\left(\sum_{i=1}^{m} a_{i}\right)+\left(\sum_{j=1}^{n} b_{i} \cdot t_{\hat{\mathcal{C}}}\left(w_{j}, w\right)\right)\right)^{\circledast}\right. \\
&\left.\left(\left(\sum_{i=1}^{p} c_{i}\right)+\left(\sum_{j=1}^{q} d_{j} \cdot t_{\hat{\mathcal{C}}}\left(u_{j}, v\right)\right)\right)\right),
\end{aligned}
$$

provided that $w$ has loop-entry transitions $\left\{w \xrightarrow{a_{i}}\left[\alpha_{i}\right] w \mid\right.$ $i=1, \ldots, m\} \cup\left\{w \xrightarrow{b_{j}}\left[\beta_{j}\right] w_{j} \mid j=1, \ldots, n \wedge w_{j} \neq w\right\}$ and body transitions $\left\{w \xrightarrow{c_{i}}\right.$ bo $\left.v \mid i=1, \ldots, p\right\} \cup\left\{w \xrightarrow{d_{j}}\right.$ bo $u_{j} \mid$ $\left.j=1, \ldots, q \wedge u_{j} \neq v\right\}$. Hereby the induction proceeds on $\left.\left.\langle | v\right|_{\text {en }},\|w\|_{\text {bo }}\right\rangle$ with the lexicographic order $<_{\text {lex }}$ on $\mathbb{N} \times \mathbb{N}$ : For $t_{\hat{C}}\left(w_{j}, w\right)$ we have $\left.\left.\left.\langle | w\right|_{\mathrm{en}},\left\|w_{j}\right\|_{\mathrm{bo}}\right\rangle<\left._{\text {lex }}\langle | v\right|_{\mathrm{en}},\|w\|_{\mathrm{bo}}\right\rangle$ due to $\left|w_{j}\right|_{\text {en }}<|v|_{\text {en }}$, which follows from $v \frown w$ by Lem. 5.2, (ii). For $t_{\hat{C}}\left(u_{j}, v\right)$ we have $\left.\left.\left.\langle | v\right|_{\text {en }},\left\|u_{j}\right\|_{\text {bo }}\right\rangle<\left._{l e x}\langle | v\right|_{\text {en }},\|w\|_{\text {bo }}\right\rangle$ due to $\left\|u_{j}\right\|_{\mathrm{bo}}<\|w\|_{\mathrm{bo}}$, which follows from $w \rightarrow_{\mathrm{bo}} u_{j}$ by Lem 5.2, (i).

The extraction function of $\hat{C}$ is defined by:

$$
\begin{aligned}
s_{\hat{C}} & : V \backslash\{\sqrt{ }\}
\end{aligned} \rightarrow \operatorname{StExp}(A), \quad \begin{aligned}
s_{\hat{C}}(w):= & \left(\left(\sum_{i=1}^{m} a_{i}\right)+\left(\sum_{j=1}^{n} b_{j} \cdot t_{\hat{C}}\left(w_{j}, w\right)\right)\right)^{\circledast} \\
& \left.\left(\left(\sum_{i=1}^{p} c_{i}\right)+\left(\sum_{j=1}^{q} d_{j} \cdot s_{\hat{C}}\left(u_{j}\right)\right)\right)\right),
\end{aligned}
$$

with induction on $\|w\|_{\text {bo }}$, provided that $w$ has loop-entry transitions $\left\{w \xrightarrow{a_{i}}{ }_{\left[\alpha_{i}\right]} w \mid i=1, \ldots, m\right\} \cup\left\{w \xrightarrow{b_{j}}{ }_{\left[\beta_{j}\right]} w_{j} \mid\right.$
$\left.j=1, \ldots, n \wedge w_{j} \neq w\right\}$ and body transitions $\left\{w \xrightarrow{c_{i}}\right.$ bo $\sqrt{ } \mid$ $i=1, \ldots, p\} \cup\left\{w \xrightarrow{d_{j}}\right.$ bo $\left.u_{j} \mid j=1, \ldots, q \wedge u_{j} \neq \sqrt{ }\right\}$. For ${ }_{s_{\hat{C}}}\left(u_{j}\right)$ the induction hypothesis holds due to $\left\|u_{j}\right\|_{\text {bo }}<\|w\|_{\text {bo }}$, which follows from $w \rightarrow_{\text {bo }} u_{j}$ by Lem. 5.2, (i).
Lemma 5.4 (uses the BBP-axioms (B1)-(B6), (BKS2), but not the rule $\left.\mathrm{RSP}^{\circledast}\right)$. In a chart $C$ with LLEE-witness $\hat{C}$, if $v \frown w$, then $s_{\hat{C}}(w)={ }_{\text {BBP }} t_{\hat{C}}(w, v) \cdot s_{\hat{C}}(v)$.
Proposition 5.5 (uses the BBP-axioms (B1)-(B6), (BKS1), (BKS2), but not the rule RSP ${ }^{\circledast}$ ). For every LLEE-witness $\hat{C}$ of a chart $C$, the extraction function $s_{\hat{C}}$ is a provable solution of $C$.

The proof of Lem. 5.4 proceeds by induction on $\|w\|_{\text {bo }}$; no induction is needed for the proof of Prop. 5.5 (cf. appendix).
Example 5.6. Left in Fig. 2 we illustrate the extraction of a provable solution for the LLEE-witness $\hat{C}=\widehat{C\left(e_{0}\right)}$ in Ex. 3.6 of the chart $C=C\left(e_{0}\right)$ in Ex. 2.5. In order to obtain the principal value $s_{\hat{C}}\left(v_{0}\right)$ of the extracted solution $s_{\hat{C}}$, its definition is expanded. It recurs on $s_{\hat{\mathcal{C}}}\left(v_{1}\right)$, and then on $t_{\hat{\mathcal{C}}}\left(v_{0}, v_{1}\right)$ and $t_{\hat{C}}\left(v_{2}, v_{1}\right)$. After computing those star expressions by using the definition of $t_{\hat{C}}$, the principal value can be obtained by substitution. The star expressions $s_{\hat{C}}\left(v_{1}\right)$ and $s_{\hat{C}}\left(v_{2}\right)$ are obtained similarly. For readability we have simplified the arising terms on the way by using the equality $0^{\circledast} x={ }_{\text {BBP }} x$ (which follows by (B1), (B6), (B7), and (BKS1)).

Lemma 5.7 (uses the BBP-axioms (B1)-(B6), and the rule RSP $\left.{ }^{\circledast}\right)$. If $v \frown w$, then $s(w)={ }_{\text {BBP }} t_{\hat{C}}(w, v) \cdot s(v)$ for every provable solution s of a chart $C$ with LLEE-witness $\hat{C}$.

Proposition 5.8 (uses the BBP-axioms (B1)-(B6), and the rule $\left.\mathrm{RSP}^{\circledast}\right)$. Let $s_{1}$ and $s_{2}$ be provable solutions of a LLEE-chart. Then $s_{1}(w)={ }_{\text {BBP }} s_{2}(w)$ for all vertices $w \neq \sqrt{ }$.

For the proof of this proposition, see Fig. 3. The proof of Lem. 5.7 (see in the appendix) proceeds by the same induction measure as we used for the relative extraction function.

Example 5.9. In the right half of Fig. 2 we prove that an arbitrary provable solution $s$ of LLEE-chart $C=\mathcal{C}\left(e_{0}\right)$ in Ex. 2.5 with LLEE-witness $\hat{C}=\widehat{C\left(e_{0}\right)}$ in Ex. 3.6 is provably equal to the extracted solution $s_{\hat{C}}$ of $C$. Crucially, the defining conditions for $s$ as a provable solution of $C$ are expanded along the loop at $v_{1}$. The loop behavior obtained is the same as that which is used in the definition of $s_{\hat{C}}\left(v_{1}\right)$. By applying the fixed-point rule RSP ${ }^{\circledast}$ we can then deduce BBP-provable equality of $s\left(v_{1}\right)$ and $s_{\hat{\mathcal{C}}}\left(v_{1}\right)$. By using the solution conditions for $s$ again, provable equality is then transferred to $v_{0}$ and $v_{1}$.

## 6 Preservation of LLEE under collapse

In this section we establish the remaining result (C) from Sect. 4 that is crucial for the completeness proof: that the bisimulation collapse of a LLEE-chart is again a LLEE-chart.

This result is achieved by a step-wise construction of a bisimulation collapse. Pairs of bisimilar vertices $w_{1}$ and $w_{2}$

$$
\begin{aligned}
& s_{\hat{C}}\left(v_{0}\right):=0^{\circledast}\left(a \cdot s_{\hat{C}}\left(v_{1}\right)\right) \\
& ={ }_{\text {BBP }} a \cdot s_{\hat{C}}\left(v_{1}\right) \\
& ={ }_{\text {BBP }} \quad a \cdot(c \cdot a+a \cdot(b+b \cdot a))^{\circledast} 0 \\
& s_{\hat{C}}\left(v_{1}\right):=\left(c \cdot t_{\hat{C}}\left(v_{0}, v_{1}\right)+a \cdot t_{\hat{C}}\left(v_{2}, v_{1}\right)\right)^{\circledast} 0 \\
& =\text { BBP }(c \cdot a+a \cdot(b+b \cdot a))^{\circledast} 0 \\
& t_{\hat{C}}\left(v_{0}, v_{1}\right):=\quad 0^{\circledast} a \\
& ={ }_{\text {BBP }} a \\
& t_{\hat{\mathcal{C}}}\left(v_{2}, v_{1}\right):=\quad 0^{\circledast}\left(b+b \cdot t_{\hat{\mathcal{C}}}\left(v_{0}, v_{1}\right)\right) \\
& ={ }_{\text {BBP }} \quad b+b \cdot a \\
& s_{\hat{C}}\left(v_{2}\right):=\quad 0^{\circledast}\left(b \cdot s_{\hat{C}}\left(v_{1}\right)+b \cdot s_{\hat{C}}\left(v_{0}\right)\right) \\
& ={ }_{\text {BBP }} b \cdot s_{\hat{C}}\left(v_{1}\right)+b \cdot\left(a \cdot s_{\hat{C}}\left(v_{1}\right)\right) \\
& ={ }_{\text {BBP }}(b+b \cdot a) \cdot s_{\hat{C}}\left(v_{1}\right) \\
& =\text { BBP }(b+b \cdot a) \cdot\left((c \cdot a+a \cdot(b+b \cdot a))^{{ }^{*}} 0\right) \\
& s\left(v_{0}\right)={ }_{\mathrm{BBP}}^{(\mathrm{sol})} a \cdot s\left(v_{1}\right) \quad\left(\begin{array}{l}
(\mathrm{sol}) \\
\text { use of } \left.{ }^{\text {means }} \text { is provable solution') }\right)
\end{array}\right. \\
& s\left(v_{1}\right)={ }_{\mathrm{BBP}}^{(\mathrm{sol})} c \cdot s\left(v_{0}\right)+a \cdot s\left(v_{2}\right) \\
& ={ }_{\mathrm{BBP}}^{(\mathrm{sol})} c \cdot\left(a \cdot s\left(v_{1}\right)\right)+a \cdot\left(b \cdot s\left(v_{1}\right)+b \cdot s\left(v_{0}\right)\right) \\
& ==_{\text {BBP }}^{\text {(sol) }} c \cdot\left(a \cdot s\left(v_{1}\right)\right)+a \cdot\left(b \cdot s\left(v_{1}\right)+b \cdot\left(a \cdot s\left(v_{1}\right)\right)\right) \\
& ={ }_{\text {BBP }}(c \cdot a+a \cdot(b+b \cdot a)) \cdot s\left(v_{1}\right)+0 \\
& \Downarrow \text { applying RSP }{ }^{\circledast} \\
& s\left(v_{1}\right)=\text { BBP }(c \cdot a+a \cdot(b+b \cdot a))^{\circledast_{0}} \\
& =\text { BBP } s_{\hat{C}}\left(v_{1}\right) \quad \text { (see in the derivation on the left) }
\end{aligned}
$$

$$
\begin{aligned}
& s\left(v_{2}\right)={ }_{\text {BBP }}^{\text {(sol) }} b \cdot s\left(v_{1}\right)+b \cdot s\left(v_{0}\right) \\
& ={ }_{\text {BBP }} b \cdot s_{\hat{C}}\left(v_{1}\right)+b \cdot s_{\hat{C}}\left(v_{0}\right)={ }_{\text {BBP }}^{(\text {sol })} s_{\hat{C}}\left(v_{2}\right)
\end{aligned}
$$

Figure 2. Left: the process of extracting the provable solution $s_{\hat{C}}$ of a chart $C$ from an LLEE-witness $\hat{C}$ of $C$ as in the middle. Right: steps for showing that an arbitrary provable solution $s$ of $C$ is BBP-provably equal to the extracted solution $s_{\hat{C}}$.

Proof (of Prop. 5.8). Let $\hat{C}$ be a LLEE-witness of a chart $C$. Let $s$ be a provable solution of $C$. We have to show that $s(w)={ }_{\text {BBP }}$ $s_{\hat{\mathcal{C}}}(w)$ for all $w \neq \sqrt{ }$. For this, let $w \neq \sqrt{ }$. The derivation below is based on the set representation of transitions from $w$ in $\hat{\mathcal{C}}$ as formulated in the definition of $s_{\hat{C}}(w)$. The first derivation step uses that $s$ is a provable solution of $C$ and axioms (B1), (B2), and (B3), the second step uses Lem. 5.7 in view of $w \frown w_{j}$ for $j=1, \ldots, n$, and the third step uses axioms (B4), (B5), and (B6).

$$
\begin{aligned}
s(w) & ={ }_{\mathrm{BBP}}\left(\left(\sum_{i=1}^{m} a_{i} \cdot s(w)\right)+\left(\sum_{j=1}^{n} b_{j} \cdot s\left(w_{j}\right)\right)\right)+\left(\left(\sum_{i=1}^{p} c_{i}\right)+\left(\sum_{j=1}^{q} d_{j} \cdot s\left(u_{j}\right)\right)\right) \\
& ={ }_{\mathrm{BBP}}\left(\left(\sum_{i=1}^{m} a_{i} \cdot s(w)\right)+\left(\sum_{j=1}^{n} b_{j} \cdot\left(t_{\hat{C}}\left(w_{j}, w\right) \cdot s(w)\right)\right)\right)+\left(\left(\sum_{i=1}^{p} c_{i}\right)+\left(\sum_{j=1}^{q} d_{j} \cdot s\left(u_{j}\right)\right)\right) \\
& ={ }_{\mathrm{BBP}}\left(\left(\sum_{i=1}^{m} a_{i}\right)+\left(\sum_{j=1}^{n}\left(b_{j} \cdot t_{\hat{C}}\left(w_{j}, w\right)\right)\right)\right) \cdot s(w)+\left(\left(\sum_{i=1}^{p} c_{i}\right)+\left(\sum_{j=1}^{q} d_{j} \cdot s\left(u_{j}\right)\right)\right)
\end{aligned}
$$

In view of this derived provable equality for $s(w)$, we can now apply the rule RSP ${ }^{*}$ in order to obtain:

$$
s(w)=\operatorname{BBP}\left(\left(\sum_{i=1}^{m} a_{i}\right)+\left(\sum_{j=1}^{n} b_{j} \cdot t_{\hat{\mathcal{C}}}\left(w_{j}, w\right)\right)\right)^{\circledast}\left(\left(\sum_{i=1}^{p} c_{i}\right)+\left(\sum_{j=1}^{q} d_{j} \cdot s_{\hat{C}}\left(u_{j}\right)\right)\right) \equiv s_{\hat{C}}(w)
$$

In this last step we have used the definition of $s_{\hat{C}}(w)$.
Figure 3. Proof of Prop. 5.8.
are collapsed one at a time, whereby the incoming transitions of $w_{1}$ are redirected to $w_{2}$. The crux is to take care, and to prove, that the resulting chart has again a LLEE-witness.

Definition 6.1. Let $C$ be a chart, with vertices $w_{1}$ and $w_{2}$.
The connect- $w_{1}$-through-to- $w_{2}$ chart $C_{w_{2}}^{\left(w_{1}\right)}$ of $C$ is obtained by redirecting all incoming transitions at $w_{1}$ over to $w_{2}$, and, if $w_{1}$ is the start vertex of $C$, making $w_{2}$ the new start vertex; in this way $w_{1}$ gets unreachable, and it is removed with other unreachable vertices to obtain a start-vertex connected chart.

Let $\hat{C}$ be an entry/body-labeling of $C$. Then we define the entry/body-labeling $\hat{C}_{w_{2}}^{\left(w_{1}\right)}$ of $C_{w_{2}}^{\left(w_{1}\right)}$ as follows: every transition in $C_{w_{2}}^{\left(w_{1}\right)}$ that was already a transition $\tau$ in $C$ inherits its marking label from $\tau$ in $\hat{C}$; and every transition in $C_{w_{2}}^{\left(w_{1}\right)}$ that arises as the redirection $\tau_{w_{2}}$ to $w_{2}$ of a transition $\tau$ to $w_{1}$ in $C$ such that $\tau_{w_{2}}$ does not coincide with a transition already in $C$ inherits its marking label from $\tau$ in $\hat{C}$.

Lemma 6.2. If $w_{1} \leftrightarrows w_{2}$ in $C$, then $C_{w_{2}}^{\left(w_{1}\right)} \leftrightarrows C$.

While the connect-through operation of bisimilar vertices in a chart thus results in a bisimilar chart, its application to a LLEE-witness (an entry/body-labeling) does not need to yield a LLEE-witness again: the property LEE may be lost.
Example 6.3. Consider the LLEE-witness $\hat{C}$ in the middle below. The unspecified action labels are assumed to facilitate that $w_{1}$ and $w_{2}$ are bisimilar. Hence also $\widehat{w}_{1}$ and $\widehat{w}_{2}$ are bisimilar. Bisimilarity is indicated by the broken lines. The connect- $w_{1}$-through-to- $w_{2}$ chart on the left is not a LLEEchart, because it does not satisfy LEE: after the loop subchart induced by the downwards transition from $\widehat{w}_{2}$ is eliminated, and garbage collection is done, the remaining chart without the dotted transitions still has an infinite path; yet it does not contain another loop subchart, because each infinite path can reach $\sqrt{ }$ without returning to its source. An example of this is the red path from $\widehat{w}_{1}$ via $w_{2}$ and $\widehat{w}_{2}$ to $\sqrt{ }$. In $\hat{C}$, the bisimilar pair $w_{1}, w_{2}$ progresses to the bisimilar pair $\widehat{w}_{1}, \widehat{w}_{2}$. The connect- $\widehat{w}_{1}$-through-to- $\widehat{w}_{2}$ chart on the right is a LLEE-chart, as witnessed by the entry/body-labeling $\hat{C}_{\widehat{w}_{2}}^{\left(\widehat{w}_{1}\right)}$.


This illustrates that bisimilar pairs of vertices must be selected carefully, to safeguard that the connect-through construction preserves LLEE. The proposition below expresses that a pair of distinct bisimilar vertices can always be selected in one of three mutually exclusive categories. Later, three LLEE-preserving transformations I, II, and III will be defined for each of these categories.

Proposition 6.4. If a LLEE-chart $C$ is not a bisimulation collapse, then it contains a pair of bisimilar vertices $w_{1}, w_{2}$ that satisfy, for a LLEE-witness of $C$, one of the conditions:
(C1) $\neg\left(w_{2} \rightarrow^{*} w_{1}\right) \wedge\left(\neg w_{1} \Rightarrow w_{2}\right.$ is not normed $)$,
(C2) $w_{2} G^{+} w_{1}$,
(C3) $\exists v \in V\left(w_{1}{ }_{d} G v \wedge w_{2} G^{+} v\right) \wedge \neg\left(w_{2} \rightarrow_{\text {bo }}^{*} w_{1}\right)$.
Condition (C1) requires that $w_{1}$ and $w_{2}$ are in different scc's, as there is no path from $w_{2}$ to $w_{1}$. The additional proviso in (C1) constrains the pair in such a way that if both are normed, then $w_{1}$ must be outside of all loops (otherwise the connect- $w_{1}$-through-to- $w_{2}$ operation does not preserve LLEE-charts, see Ex. 6.3); its asymmetric formulation helps to avoid the assumption of bisimilarity in Prop. 6.8 below. The
two other conditions concern the situation that $w_{1}$ and $w_{2}$ are in the same scc. While in (C2) $w_{1}$ and $w_{2}$ are comparable (but different) by the loops-back-to relation $G^{*}$, they are incomparable in (C3). In the situation that $w_{1}, w_{2}$ loop back to the same vertex $v$, but $w_{1}$ directly loops back to $v$, (C3) also demands that no body step path exists from $w_{2}$ to $w_{1}$ (otherwise the connect- $w_{1}$-through-to- $w_{2}$ construction does not preserve LLEE-charts, see an example in the appendix).

In the proof of Prop. 6.4 we progress, from a given pair of distinct bisimilar vertices, repeatedly via transitions, at one side picking loop-back transitions, over pairs of distinct bisimilar vertices, until one of the conditions (C1), (C2), (C3) is met. We will use a subset of the body transitions in a LLEE-witness. By a loop-back transition, written as $u \rightarrow_{l b} v$, we mean a transition $u \rightarrow_{\text {bo }} v$ that stays within an scc, that is, $\operatorname{scc}(u)=\operatorname{scc}(v)$. The loops-back-to norm $\|u\|_{l b}^{\min }$ of $u$ is the maximal length of $\mathrm{a} \rightarrow_{l b}$ path from $u$ (which is well-defined by Lem. 3.9, (i) and chart finiteness). Note that $\|u\|_{l b}^{\min }=0$ if and only if $u$ does not loop back (denoted by $\neg(u G)$ ).

Proof of Prop. 6.4. We pick distinct bisimilar vertices $u_{1}, u_{2}$. First we consider the case $\operatorname{scc}\left(u_{1}\right) \neq \operatorname{scc}\left(u_{2}\right)$. Without loss of generality, suppose $\neg\left(u_{2} \rightarrow^{*} u_{1}\right)$. We progress to a pair of vertices where (C1) holds, using induction on $\left\|u_{1}\right\|_{l b}^{\min }$. In the base case, $\left\|u_{1}\right\|_{l b}^{\min }=0$, it suffices to show that it is not possible that both $\frown u_{1}$ holds and $u_{2}$ is normed, because then we can define $w_{1}=u_{1}$ and $w_{2}=u_{2}$, and are done. Therefore suppose, toward a contradiction, that $\frown u_{1}$ holds and $u_{2}$ is normed. Then $u_{1}$ is normed, too, since $u_{1}$ and $u_{2}$ are bisimilar. Also $\neg\left(u_{1} G\right)$ follows from $\left\|u_{1}\right\|_{l b}^{\min }=0$, which says that there are no loops-back-to steps from $u_{1}$. So we get that $\frown u_{1}, \neg\left(u_{1} G\right)$, and $u_{1}$ is normed. This contradicts Lemma 3.9, (iii). In the induction step, $\left\|u_{1}\right\|_{l b}^{\min }>0$ implies $u_{1} \rightarrow_{l b} u_{1}^{\prime}$ and $\left\|u_{1}^{\prime}\right\|_{l b}^{\min }<\left\|u_{1}\right\|_{l b}^{\min }$ for some $u_{1}^{\prime}$. Since $u_{1} \leftrightarrow u_{2}$, we have $u_{2} \rightarrow u_{2}^{\prime}$ and $u_{1}^{\prime} \leftrightarrows u_{2}^{\prime}$ for some $u_{2}^{\prime}$. Since $u_{1} \rightarrow{ }_{l b} u_{1}^{\prime}$, by definition, $u_{1}$ and $u_{1}^{\prime}$ are in the same scc. Hence $u_{1}^{\prime} \rightarrow^{*} u_{1}$. This implies $\neg\left(u_{2}^{\prime} \rightarrow^{*} u_{1}^{\prime}\right)$, for else $u_{2} \rightarrow u_{2}^{\prime} \rightarrow^{*} u_{1}^{\prime} \rightarrow^{*} u_{1}$, which contradicts the assumption $\neg\left(u_{2} \rightarrow^{*} u_{1}\right)$. Since $u_{1}^{\prime} \leftrightarrows$ $u_{2}^{\prime}$ and $\neg\left(u_{2}^{\prime} \rightarrow^{*} u_{1}^{\prime}\right)$ and $\left\|u_{1}^{\prime}\right\|_{l b}^{\min }<\left\|u_{1}\right\|_{l b}^{\min }$, by induction there exists a bisimilar pair $w_{1}, w_{2}$ for which (C1) holds.

Now let $\operatorname{scc}\left(u_{1}\right)=\operatorname{scc}\left(u_{2}\right)$. Then by Lem. 3.9, (iv), $u_{1} G^{*} v$ and $u_{2} G^{*} v$ for some $v$. By Lem. 3.9, (v) we pick $v$ as the least upper bound of $u_{1}, u_{2}$ with regard to $G^{*}$. If $u_{1}=v$, then $u_{2} G^{+} u_{1}$, so (C2) holds for $w_{1}=u_{1}$ and $w_{2}=u_{2}$. If $u_{2}=v$, then likewise (C2) holds for $w_{1}=u_{2}$ and $w_{2}=u_{1}$. Now let $u_{1}, u_{2} \neq v$. Since $v$ is the least upper bound, $u_{1} G^{*} v_{1}{ }_{d} G$ $v_{d} \bigcirc v_{2} \bigcirc^{*} u_{2}$ for distinct $v_{1}, v_{2} \in V$. There cannot be a cycle of body transitions, so $\neg\left(v_{2} \rightarrow_{\text {bo }}^{*} v_{1}\right)$ or $\neg\left(v_{1} \rightarrow_{\text {bo }}^{*} v_{2}\right)$. By symmetry it suffices to consider $\neg\left(v_{2} \rightarrow_{\text {bo }}^{*} v_{1}\right)$. Summarizing, $u_{1} G^{*} v_{1}{ }_{d} G v_{d} \bigcirc v_{2} \bigcirc^{*} u_{2}$ and $\neg\left(v_{2} \rightarrow{ }_{\text {bo }}^{*} v_{1}\right)$. For this situation we use induction on $\left\|u_{1}\right\|_{l b}^{\min }$. If $u_{1}=v_{1}$, then $u_{1}{ }_{d} G v$; taking $w_{1}=u_{1}$ and $w_{2}=u_{2}$, (C3) holds. So we can assume $u_{1} G^{+} v_{1}{ }_{d} G v$. Pick a transition $u_{1} \rightarrow l b u_{1}^{\prime}$ with
$\left\|u_{1}^{\prime}\right\|_{l b}^{\min }<\left\|u_{1}\right\|_{l b}^{\min }$; by definition, $\operatorname{scc}\left(u_{1}^{\prime}\right)=\operatorname{scc}\left(u_{1}\right)$. Since $u_{1} \leftrightarrows u_{2}$, there is a transition $u_{2} \rightarrow u_{2}^{\prime}$ with $u_{1}^{\prime} \leftrightarrows u_{2}^{\prime}$ for some $u_{2}^{\prime}$. If $\operatorname{scc}\left(u_{1}^{\prime}\right) \neq \operatorname{scc}\left(u_{2}^{\prime}\right)$, then as before we can find bisimilar $w_{1}, w_{2}$ for which (C1) holds. Now let $\operatorname{scc}\left(u_{1}^{\prime}\right)=\operatorname{scc}\left(u_{2}^{\prime}\right)$, so $u_{1}, u_{2}, u_{1}^{\prime}, u_{2}^{\prime}$ are in the same scc. Since $u_{1} G^{+} v_{1}$ and $u_{1} \rightarrow u_{1}^{\prime}$, either $u_{1}^{\prime}=v_{1}$ or $v_{1} \frown^{+} u_{1}^{\prime}$. Moreover, $\operatorname{scc}\left(u_{1}^{\prime}\right)=\operatorname{scc}\left(u_{1}\right)=$ $\operatorname{scc}\left(v_{1}\right)$, so by Lem. 3.9, (ii), $u_{1}^{\prime} G^{*} v_{1}$. Since $u_{2} G^{*} v_{2}$, we can distinguish two cases (for illustrations for each of the subcases, see the appendix).
Case 1: $u_{2} G^{+} v_{2}$. Since $u_{2} \rightarrow u_{2}^{\prime}$, either $u_{2}^{\prime}=v_{2}$ or $v_{2} \frown^{+} u_{2}^{\prime}$. Moreover, $\operatorname{scc}\left(u_{2}^{\prime}\right)=\operatorname{scc}\left(u_{2}\right)=\operatorname{scc}\left(v_{2}\right)$, so by Lem. 3.9, (ii), $u_{2}^{\prime} G^{*} v_{2}$. Hence, $u_{1}^{\prime} G^{*} v_{1}{ }_{d} G v_{d} \bigcirc$ $v_{2} \frown^{*} u_{2}^{\prime} \wedge \neg\left(v_{2} \rightarrow_{\text {bo }}^{*} v_{1}\right)$, and $\left\|u_{1}^{\prime}\right\|_{l b}^{\min }<\left\|u_{1}\right\|_{l b}^{\min }$. We apply the induction hypothesis to obtain a bisimilar pair $w_{1}, w_{2}$ for which (C1), (C2), or (C3) holds. Below we illustrate both of the cases in which $u_{2} \rightarrow u_{2}^{\prime}$ is a loop-entry transition, or a body transition.


Case 2: $u_{2}=v_{2}$. We distinguish two cases.
Case 2.1: $u_{2} \rightarrow{ }_{[\alpha]} \quad u_{2}^{\prime}$. Then either $u_{2}^{\prime}=u_{2}$ or $u_{2} \frown^{+}$ $u_{2}^{\prime}$. Moreover, $\operatorname{scc}\left(u_{2}^{\prime}\right)=\operatorname{scc}\left(u_{2}\right)$, so by Lem. 3.9, (ii), $u_{2}^{\prime} G^{*} u_{2}$, and hence $u_{2}^{\prime} G^{*} v_{2}$. Thus we have obtained $u_{1}^{\prime} G^{*} v_{1}{ }_{d} G v_{d} \triangleright v_{2} \bigcirc^{*} u_{2}^{\prime} \wedge \neg\left(v_{2} \rightarrow_{\text {bo }}^{*} v_{1}\right)$. Due to $\left\|u_{1}^{\prime}\right\|_{l b}^{\min }<\left\|u_{1}\right\|_{l b}^{\min }$, we can apply the induction hypothesis again.
Case 2.2: $u_{2} \rightarrow_{\text {bo }} u_{2}^{\prime}$. Then $\neg\left(v_{2} \rightarrow_{\text {bo }}^{*} v_{1}\right)$ together with $v_{2}=u_{2} \rightarrow_{\text {bo }} u_{2}^{\prime}$ and $u_{1}^{\prime} \rightarrow_{\text {bo }}^{*} v_{1}$ (because $u_{1}^{\prime} G^{*} v_{1}$ ) imply $u_{1}^{\prime} \neq u_{2}^{\prime}$. We distinguish two cases.
Case 2.2.1: $u_{2}^{\prime}=v$. Then $u_{1}^{\prime} G^{*} v_{1}{ }_{d} G v=u_{2}^{\prime}$, i.e., $u_{1}^{\prime} G^{+}$ $u_{2}^{\prime}$, so we are done, because (C2) holds for $w_{1}=u_{2}^{\prime}$ and $w_{2}=u_{1}^{\prime}$.
Case 2.2.2: $u_{2}^{\prime} \neq v$. By Lem. 3.9, (ii), $u_{2}^{\prime} G^{+} v$. Hence, $u_{2}^{\prime} G^{*}$ $v_{2}^{\prime}{ }_{d} \mathrm{G} v$ for some $v_{2}^{\prime}$. Since $v_{2}=u_{2} \rightarrow$ bo $u_{2}^{\prime} \mathrm{G}^{*}$ $v_{2}^{\prime}$ and $\neg\left(v_{2} \rightarrow_{\text {bo }}^{*} v_{1}\right)$, it follows that $\neg\left(v_{2}^{\prime} \rightarrow_{\text {bo }}^{*} v_{1}\right)$. So $u_{1}^{\prime} G^{*} v_{1}{ }_{d} G v_{d} \bigcirc v_{2}^{\prime} \frown^{*} u_{2}^{\prime} \wedge \neg\left(v_{2} \rightarrow{ }_{\text {bo }}^{*} v_{1}^{\prime}\right)$. Due to $\left\|u_{1}^{\prime}\right\|_{l b}^{\min }<\left\|u_{1}\right\|_{l b}^{\min }$, we can apply the induction hypothesis again.

This exhaustive case analysis concludes the proof.

Now we define, for LLEE-witnesses $\hat{C}$ of a LLEE-chart $\mathcal{C}$, and for bisimilar vertices $w_{1}, w_{2}$ in $C$, in each of the three cases (C1), (C2) , or (C3) of Prop. 6.4 a transformation of $\hat{C}$ into an entry/body-labeling of the connect- $w_{1}$-through-to$w_{2} \operatorname{chart} C_{w_{2}}^{\left(w_{1}\right)}$ that can be shown to be a LLEE-witness again. We number the transformations for (C1), (C2) , and (C3) as I, II, and III, respectively. Each transformation makes use of the connect-through construction for entry/body-labelings as defined in Def. 6.1. Additionally, in each transformation an adaptation of labels of transitions is performed, to avoid violations of LLEE-witness properties. In transformations I and III the adaptation is performed before connecting $w_{1}$ through to $w_{2}$, and is needed to guarantee that layeredness is preserved; in transformation II it is performed right after eliminating $w_{1}$, and avoids the creation of body step cycles. The level adaptations for the three transformations are:
$L_{\mathrm{I}}$ Let $m=\max \left\{\beta:\right.$ there is a path $w_{2} \rightarrow^{*} \cdot \rightarrow_{[\beta]}$ in $\left.\hat{C}\right\}$. In loop-entry transitions $u \rightarrow[\alpha]$ for which there is a path $v \rightarrow^{*} w_{1}$ in $C$, replace $\alpha$ by an $\alpha^{\prime}$ with $\alpha^{\prime}=\alpha+m$. This increases the labels of loop-entry transitions that descend to $w_{1}$ in $\hat{C}$ to a higher level than the loop labels reachable from $w_{2}$.
$L_{\mathrm{II}}$ Since $w_{2} \mathrm{G}^{+}{ }^{1} w_{1}$, there exists a $\widehat{w}_{2}$ with $w_{2} \mathrm{G}^{*} \widehat{w}_{2}{ }_{d} \mathrm{G}$ $w_{1}$. Let $\gamma$ be the maximum loop level among the loopentries at $w_{1}$ in $\hat{C}$. (Note that since $w_{2} G^{+} w_{1}$, there is at least one such transition.) Turn the body transitions from $\widehat{w}_{2}$ into loop-entry transitions with loop label $\gamma$.
$L_{\text {III }}$ Let $\gamma$ be a loop label of maximum level among the loopentry transitions at $v$ in $\hat{\mathcal{C}}$. (Note that since $w_{1} G v$, there is at least one such transition.) Turn the loop labels of the loop-entry transitions from $v$ into $\gamma$.

Each of these transformations ends with a clean-up step: if the loop-entry transitions from a vertex with the same loop label no longer induce an infinite path (due to the removal of $w_{1}$ ), then they are changed into body transitions.

Example 6.5. The LLEE-witness on the left in Fig. 4 is reduced in three transformation steps to a LLEE-witness of the chart $\mathcal{C}\left(e_{0}\right)$ in Ex. 2.5. Broken lines are between bisimilar vertices. In step one, a transformation I, the start state $v_{0}$ is connected through to the bisimilar vertex $v_{0}^{\prime \prime}$, whereby $v_{0}^{\prime \prime}$ becomes the start vertex; note that there is no path from $v_{0}^{\prime \prime}$ to $v_{0}$, and no vertex descends into a loop to $v_{0}$. In step two, a transformation II, $v_{1}$ is connected through to the bisimilar vertex $v_{1}^{\prime}$; note that $v_{1}^{\prime} G^{+} v_{1}$. In step three, a transformation III, the start vertex $v_{0}^{\prime \prime}$ is connected through to the bisimilar vertex $v_{0}^{\prime \prime \prime}$, whereby $v_{0}^{\prime \prime \prime}$ becomes the start vertex; note that $v_{0}^{\prime \prime}{ }_{d} G v_{2}$ and $v_{0}^{\prime \prime \prime} G^{+} v_{2}$ and there is no body step path from $v_{0}^{\prime \prime \prime}$ to $v_{0}^{\prime \prime}$. By the loop level adaptation $L_{\text {III }}$, all loop entries from $v_{2}$ get level 3. The final step is an isomorphic deformation. Only the left and right charts depict actions.


Figure 4. Three connect-through-steps according to the transformations I, II, and III from the LLEE-witness on the left, and a final isomorphic deformation, leading to the LLEE-witness on the right. For clarity, we neglected action labels in the middle.

The following examples provide more illustrations of the transformations II and III. Similarly as Ex. 6.3 does so for transformation I and (C1), they also show that the conditions (C2) and (C3) mark rather sharp borders between whether, on a given LLEE-witness, a connect-through operation is possible while preserving LLEE, or not.
Example 6.6. For the LLEE-witness $\hat{C}$ below in the middle, the chart $C_{w_{1}}^{\left(w_{2}\right)}$ on the left has no LLEE-witness.


It does not satisfy LEE: it has no loop subchart, since from each of its three vertices an infinite path starts that does not return to this vertex; from $\widehat{w}_{2}$ this path, drawn in red, cycles between $u$ and $w_{1}$. Transformation II applied to the pair $w_{1}, w_{2}$ (instead of $w_{2}, w_{1}$ ) in $\hat{C}$ yields the entry/body-labeling $\hat{C}_{w_{2}}^{\left(w_{1}\right)}$ where $\widehat{w}_{2} \rightarrow_{\text {bo }} w_{2}$ is turned into $\widehat{w}_{2} \rightarrow{ }_{[2]} w_{2}$. As the pair $w_{1}, w_{2}$ satisfies (C2) , the proof of Prop. 6.8 ensures that this labeling, drawn on the right, is a LLEE-witness.

Example 6.7. In the LLEE-witness $\hat{C}$ below in the middle, $w_{1}, w_{2} G^{+} v$ and there is no body step path from $w_{2}$ to $w_{1}$, but (C3) does not hold for the pair $w_{1}, w_{2}$ due to $\neg\left(w_{1}{ }_{d} G v\right)$. The chart $C_{w_{2}}^{\left(w_{1}\right)}$ on the left has no LLEE-witness. It does not satisfy LEE: the downwards loop-entry transition from $\widehat{w}_{2}$
can be eliminated, and then two more arising loop-entry transitions from $v$; the remaining chart of solid arrows has no further loop subchart, because from each of its vertices an infinite path starts that does not return to this vertex.

In $\hat{C}$, loop-entry transitions from $v$ have the same loop label, so the preprocessing step of transformation III is void. The bisimilar pair $w_{1}, w_{2}$ progresses to the bisimilar pair $\widehat{w}_{1}, \widehat{w}_{2}$ in $\hat{C}$, for which (C3) holds because $\widehat{w}_{1}{ }_{d} G v \circlearrowleft \widehat{w}_{2}$ and $\neg\left(\widehat{w}_{2} \rightarrow{ }_{\text {bo }}^{*} \widehat{w}_{1}\right)$. Transformation III applied to this pair yields the labeling $\hat{C}_{\widehat{w}_{2}}^{\left(\widehat{w}_{1}\right)}$ on the right. In the proof of Prop. 6.8 it is argued that this is guaranteed to be a LLEE-witness. The remaining two bisimilar pairs can be eliminated by one or by two further applications of transformation III.


Proposition 6.8. Let $C$ be a LLEE-chart. If a pair $\left\langle w_{1}, w_{2}\right\rangle$ of vertices satisfies (C1), (C2), or (C3) with respect to a LLEE-witness of $C$, then $C_{w_{2}}^{\left(w_{1}\right)}$ is a LLEE-chart.

Proof. Let $\hat{C}$ be a LLEE-witness. For vertices $w_{1}, w_{2}$ such that (C1) , (C2) , or (C3) holds, transformation I, II, or III, respectively, produces an entry/body-labeling $\hat{C}_{w_{2}}^{\left(w_{1}\right)}$. We prove for transformation I that this is a LLEE-witness, and refer to the appendix with regard to transformations II, and III.

We first argue it suffices to show that each of the transformations produces, before the final clean-up step, a labeling that satisfies the LLEE-witness conditions, except possible violations of loop property (L1) in (W2)(a). Such violations can be removed from a loop-labeling while preserving the other LLEE-witness conditions. To show this, suppose (L1) is violated in some $C_{\hat{\mathcal{C}}}(u, \alpha)$. Then $u \rightarrow{ }_{[\alpha]}$ but $\neg\left(u \rightarrow{ }_{[\alpha]} \rightarrow_{\text {bo }}^{*} u\right)$. Let $\hat{C}_{1}$ be the result of removing this violation by changing the $\alpha$-loop-entry transitions from $u$ into body transitions. No new violation of (L1) is introduced in $\hat{C}_{1}$. (W1) and (W2)(a), (L2), are preserved in $\hat{C}_{1}$ because an introduced infinite body step path in $\hat{\mathcal{C}}_{1}$ would be a body step cycle that stems from a path $u \rightarrow{ }_{[\alpha]} u^{\prime} \rightarrow_{\text {bo }}^{*} u$ in $\hat{C}$. (W2)(b) might only be violated by a path $w \underset{\boldsymbol{t}(w)}{\longrightarrow}[\beta] \cdot \underset{\boldsymbol{t}(w)}{ }{ }_{\text {bo }}^{*} u \underset{\boldsymbol{t ( w , u )}}{ }$ bo $u^{\prime} \xrightarrow[\mathbf{t}(w, u)]{ }{ }^{*}$ bo $\cdot{ }_{[\gamma]}$ with $\beta \leqslant \gamma$ in $\hat{C}_{1}$ where $u \rightarrow_{\text {bo }} u^{\prime}$ stems from $u \rightarrow[\alpha] u^{\prime}$ in $\hat{C}$; then $\beta>\alpha>\gamma$ by layeredness of $\hat{C}$; so (W2)(b) is preserved. Analogously we find that also (W2)(a), (L3) is preserved, because $\sqrt{ }$ is never in $C_{\hat{C}}(u, \alpha)$.

To show the correctness of transformation I, consider vertices $w_{1}$ and $w_{2}$ with (C1). We show that the result $\hat{C}_{w_{2}}^{\left(w_{1}\right)}$ of transformation I before the clean-up step satisfies the LLEEwitness properties, except for possible violations of (L1).

To verify (W1) and part (L2) of (W2)(a), it suffices to show that $\hat{C}_{w_{2}}^{\left(w_{1}\right)}$ does not contain body step cycles. The original loop-labeling $\hat{C}$ is a LLEE-witness, so it does not contain body step cycles. Since the level adaptation step does not turn loop-entry steps into body steps, body step cycles could only arise in the step connecting $w_{1}$ through to $w_{2}$. Suppose such a body step cycle arises. Then there must be a transition $u \rightarrow$ bo $w_{1}$ in $\hat{C}$ (which is redirected to $w_{2}$ in $\hat{C}_{w_{2}}^{\left(w_{1}\right)}$ ) and a path $w_{2} \rightarrow_{\text {bo }}^{*} u$ in $\hat{C}$. But then $w_{2} \rightarrow_{\text {bo }}^{*} u \rightarrow$ bo $w_{1}$ in $C$, which contradicts (C1) that there is no path from $w_{2}$ to $w_{1}$. Hence (W1) and part (L2) of (W2)(a) hold for $\hat{C}_{w_{2}}^{\left(w_{1}\right)}$.

Now we verify part (L3) of (W2)(a) in $\hat{C}_{w_{2}}^{\left(w_{1}\right)}$. Consider a path $u \underset{ \pm(u)}{\longrightarrow}[\alpha] \cdot{\underset{\mathbf{t}(u)}{ }}^{*}$ bo $w_{1}$ in $\hat{C}$. Then $u \neq w_{1}$, and $u \frown w_{1}$. It suffices to show that then $\neg\left(w_{2} \rightarrow^{+} \sqrt{ }\right)$ in $C$. But this is guaranteed, because otherwise $w_{2}$ were normed, and due to $u \leadsto w_{1}$ we would have a contradiction with condition (C1) .

Finally we show that (W2)(b) is preserved in $\hat{C}_{w_{2}}^{\left(w_{1}\right)}$ by both the level adaptation and the connect-through step. First, since in the level adaptation step all adapted loop labels are increased with the same value $m$, a violation of (W2)(b) would arise by a path $u \rightarrow[\alpha] \cdot \rightarrow_{\text {bo }}^{*} \cdot \rightarrow_{[\beta]} v$ in $\hat{C}$ where loop label $\beta$ is increased while $\alpha$ is not. But such a path cannot exist. Since $\beta$ is increased, there is a path $v \rightarrow^{*} w_{1}$ in $C$. But then there is a path $u \rightarrow[\alpha] \cdot \rightarrow^{+} v \rightarrow^{*} w_{1}$ in $\hat{\mathcal{C}}$, which implies that also $\alpha$ is increased in the level adaptation step. Second, a violation of (W2)(b) in the connect-through step would arise from paths $u \rightarrow[\alpha] \cdot \rightarrow_{\text {bo }}^{*} w_{1}$ and $w_{2} \rightarrow_{\text {bo }}^{*} \cdot \rightarrow_{[\beta]}$ in $\hat{\mathcal{C}}^{\prime}$ with $\alpha \leqslant \beta$. However, in view of the path $u \rightarrow[\alpha] \cdot \rightarrow^{*} w_{1}$,
the loop label $\alpha$ was increased with $m$ in the level adaptation step. On the other hand, in view of (C1) that there is no path from $w_{2}$ to $w_{1}$ in $C, w_{1}$ is unreachable at the end of the path $w_{2} \rightarrow^{*} \cdot \rightarrow{ }_{[\beta]}$. Hence this loop label $\beta$ was not increased in the level adaptation step. So it is guaranteed that for such a pair of paths in $\hat{C}_{w_{2}}^{\left(w_{1}\right)}$ always $\alpha>\beta$.

We conclude that the result of transformation $I$ is again a LLEE-witness.

Theorem 6.9. The bisimulation collapse of a LLEE-chart is again a LLEE-chart.

Proof. Given a LLEE-chart $\mathcal{C}$, repeat the following step: based on a LLEE-witness pick, by Prop. 6.4, bisimilar vertices $w_{1}$ and $w_{2}$ with ( C 1$)$, (C2), or (C3), and then connect $w_{1}$ through to $w_{2}$, obtaining by Prop. 6.8 a LLEE-chart bisimilar to $C$, due to Lem. 6.2. Hence the bisimulation collapse of $C$, which is reached eventually, is a LLEE-chart.

We mention that by using a refinement of the interpretation TSS (that avoids creating concatenations $e_{1} \cdot e_{2}$ where $e_{1}$ is not normed, in favor of using just $e_{1}$ ) and a refinement of the extraction procedure (that ensures an eager use of the right distributive law (B4) of - over + ) this theorem can be strengthened: the bisimulation collapse of a LLEE-chart is the chart interpretation of some star expression. which then is a LLEE-chart by Prop. 2.9. This can be proved by showing that, on collapsed LLEE-charts, (refined) chart interpretation is the converse of (refined) solution extraction.

Corollary 6.10. If a chart is expressible by a star expression modulo bisimilarity, then its collapse is a LLEE-chart.

The converse statement holds as well. But this corollary does not hold for star expressions with 1 and unary star. For example, with respect to the TSS for the process interpretation of star expressions from this class, see e.g. [3], the expression $e_{1}:=\left(\left(\left(1 \cdot a^{*}\right) \cdot\left(b \cdot c^{*}\right)\right) \cdot e\right.$ with $e:=\left(a^{*} \cdot\left(b \cdot c^{*}\right)\right)^{*}$ has the following interpretation, where $e_{2}:=\left(1 \cdot c^{*}\right) \cdot e$ :


This is a chart in the extended sense in which immediate termination is permitted at arbitrary vertices. It is a bisimulation collapse that does not satisfy LEE, taking into account that in the definition of 'loop' for charts in the extended sense (L3) needs to be changed to exclude immediate termination for vertices in a loop chart other than the start vertex.

## 7 The completeness result, and conclusion

That bisimulation collapse preserves LLEE was the last building block in the proof of the desired completeness result.

Theorem 7.1. The proof system BBP is complete with respect to the bisimulation semantics of star expressions, that is, with
respect to bisimilarity of charts that interpret star expressions without 1 and with binary Kleene star ${ }^{\circledast}$.
Proof. The proof steps were already explained in Sect. 4.
Example 7.2. The bisimilar LLEE-charts $C_{1}$ and $C_{2}$ in Ex. 4.1 have $(a \cdot(a+b)+b){ }^{\circledast} 0$ and $(b \cdot(a+b)+a)^{\circledast} 0$ as their principal solutions. Their bisimulation collapse $C_{0}$ has principal solution $(a+b)^{\circledast} 0$. Then $(a \cdot(a+b)+b)^{\circledast} 0={ }_{\text {BBP }}$ $(a+b)^{\circledast} 0={ }_{\text {BBP }}(b \cdot(a+b)+a)^{\circledast} 0$ by Prop. 5.1, Prop. 5.8.

Example 7.3. Revisiting the star expressions $e_{1}, e_{2}$ in Ex. 2.5 with bisimilar chart interpretations $C\left(e_{1}\right)$ and $C\left(e_{2}\right)$, we can apply our proof in order to show that $e_{1}={ }_{\mathrm{BBP}} e_{2} . C\left(e_{1}\right)$ and $C\left(e_{2}\right)$ have provable solutions with principal values $e_{1}$ and $e_{2}$ by Prop. 2.9. As $C\left(e_{1}\right)$ and $C\left(e_{2}\right)$ are LLEE-charts by Prop. 3.7 with LLEE-witnesses $\widehat{C\left(e_{1}\right)}$ and $\widehat{C\left(e_{2}\right)}$, their bisimulation collapse $C$ is a LLEE-chart by Thm. 6.9. We take here the more familiar $\hat{C}$, but could also take the one obtained in Fig. 4. We saw in Fig. 2 that $\hat{C}$ has a provable solution with principal value $s_{\hat{C}}\left(v_{0}\right)=a \cdot\left((c \cdot a+a \cdot(b+b \cdot a))^{\circledast} 0\right)$. Then by Prop. 5.1 and Prop. 5.8 it follows that $e_{1}={ }_{\mathrm{BBP}} s_{\hat{C}}\left(v_{0}\right)={ }_{\mathrm{BBP}} e_{2}$.


We have shown that Milner's axiomatization, tailored to star expressions without 1 and with ${ }^{\circledast}$, is complete in bisimulation semantics. At the core of our proof is the graph structure property LLEE, which characterizes the process graphs that can be expressed by star expressions without 1 and with $\circledast$ as charts whose bisimulation collapse is a LLEE-chart.

Completeness of BBP covers completeness of the theory $\mathrm{BPA}_{0}^{\omega}+\mathrm{RSP}^{\omega}$ of perpetual loop iteration $(\cdot)^{\omega}[10]$ in the sense that the latter result can be shown by our means, or by a faithful interpretation $e^{\omega} \mapsto e^{\circledast} 0$ of $\mathrm{BPA}_{0}^{\omega}+\mathrm{RSP}^{\omega}$ in BBP .

Completeness of BBP can be extended, also by means of a faithful interpretation, to cover star expressions with 0 , 1 , and $*$, but with a syntactic restriction on terms directly under a *: that they can be rewritten to star expressions with only 'harmless' occurrences of 1 . This is analogous to the situation that the completeness result from [9,11] for star expressions without 0 and 1 and with ${ }^{\circledast}$ was extended in [7] to a setting with 1 (but not 0 ) and *, where a generalized version of the non-empty-word property is disallowed for
terms directly under a *. With the interpretation approach, also the result in [7] can be obtained from the one in [9,11].

The main future goal is to solve Milner's problem entirely by extending our result to the full class of star expressions.

## Acknowledgments

We thank Alban Ponse for his suggestion to consider completeness of Milner's axiomatization for the fragment without 1, and Luca Aceto for encouragement, comments on the exposition, and facilitating a visit of the second author to GSSI, from which this paper developed. Also, we thank the reviewers for detailed remarks and suggestions on how to improve the positioning of our completeness result.

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## A Appendix: supplements, more proof details, and omitted proofs

## A. 1 Proofs in Section 2: Preliminaries

Proposition (= Proposition 2.9, uses BBP-axioms (B1)-(B7), (BKS1)). For every e $\in \operatorname{StExp}(A)$, the identity function $\mathrm{id}_{V(e)}$ : $V(e) \rightarrow V(e) \subseteq \operatorname{StExp}(A), e^{\prime} \mapsto e^{\prime}$, is a provable solution of the chart interpretation $C(e)$ of $e$.

In the proof of this proposition we will use the following definition concerning 'action derivatives', and the subsequent lemma. That statement can be viewed as the 'fundamental theorem of differential calculus for star expressions' which says that every star expressions can be reassembled by a form of 'integration' from its action derivatives. In this context 'differentiation' follows the definition of action derivatives in Definition 2.4 (corresponding to Antimirov's concept of 'partial derivative' in [2]), and 'integration' means sum formation over products of pairs $\langle a, \xi\rangle$ for actions $a$ and $a$-derivatives $\xi$.
Definition A.1. For star expressions $e \in \operatorname{StExp}(A)$ we define the set $A \partial(e)$ of action derivatives of $e$ as follows:

$$
A \partial(e):=\left\{\langle a, \xi\rangle \mid a \in A, \xi \in \operatorname{StExp}(A)_{\sqrt{ }}, e \xrightarrow{a} \xi\right\} .
$$

Lemma A.2. Every $e \in \operatorname{StExp}(A)$ can be provably reassembled from its action derivatives as:

$$
\begin{align*}
e=\mathrm{BBP} & \left(\sum_{i=1}^{m} a_{i}\right)+\left(\sum_{j=1}^{n} b_{j} \cdot e_{j}^{\prime}\right)  \tag{A.1}\\
& \quad \text { provided that A} \partial(e)=\left\{\left\langle a_{1}, \sqrt{ }\right\rangle, \ldots,\left\langle a_{m}, \sqrt{ }\right\rangle,\left\langle b_{1}, e_{1}^{\prime}\right\rangle, \ldots,\left\langle b_{n}, e_{n}^{\prime}\right\rangle\right\} \tag{A.2}
\end{align*}
$$

Proof. We start by noting that we need to show (A.1), for all $e \in \operatorname{StExp}(A)$, only for one list representation of $A \partial(e)$ of the form (A.2). This is because then (A.1) follows also for all other list representations of $A \partial(e)$ the form (A.2). Indeed, the axioms (B1), (B2), and (B3) of BBP (the ACl-axioms for associativity, commutativity, and idempotency of + ) can be used to permute and duplicate summands as well as to remove duplicates of summands in sums (A.1) according to permutations, duplications, and removal of duplicates in list representations of $A \partial(e)$ of the form (A.2).

We proceed by induction on the structure of star expressions in $\operatorname{StExp}(A)$. For performing the induction step, we distinguish the five cases of productions in the grammar in Definition 2.1.
Case 1: $e \equiv 0$.
Then $e$ does not enable any transitions, and hence $A \partial(e)=\varnothing$. We find the provable equality:

$$
e \equiv 0={ }_{\text {BBP }} 0+0 \quad \text { (by axiom (B1) of BBP) }
$$

This is of the form as in (A.1) with $m=n=0$ when we construe $A \partial(e)=\varnothing$ as a list representation of the form (A.2).
Case 2: $e \equiv a$ for some $a \in A$.
Then according to the TSS in Definition 2.4 the expression $e$ enables precisely one transition, an $a$-transition to $\sqrt{ }$. Hence the set of action derivatives of $e$ consists only of one element:

$$
\begin{equation*}
A \partial(e)=\{\langle a, \sqrt{ }\rangle\} \tag{A.3}
\end{equation*}
$$

We find the provable equality:

$$
e={ }_{\mathrm{BBP}} a+0 \quad \text { (by axiom (B6) of BBP) } .
$$

The right-hand side is of the form (A.1) with $m=1, a_{1}=a$ and $n=0$ in relation to (A.3) when we construe $A \partial(e)$ as a list representation of the form (A.2).

Case 3: $e \equiv e_{1}+e_{2}$.
Since every star expression has only finitely many derivatives, each of which is either $\sqrt{ }$ or a star expression, we may assume that the sets of action derivatives of the constituent expressions $e_{1}$ and $e_{2}$ of $e_{1}+e_{2}$ have list representations:

$$
\begin{align*}
& A \partial\left(e_{1}\right)=\left\{\left\langle a_{11}, \sqrt{ }\right\rangle, \ldots,\left\langle a_{m_{1} 1}, \sqrt{ }\right\rangle,\left\langle b_{11}, e_{11}^{\prime}\right\rangle, \ldots,\left\langle b_{n_{1} 1}, e_{n_{1} 1}^{\prime}\right\rangle\right\} \\
& A \partial\left(e_{2}\right)=\left\{\left\langle a_{12}, \sqrt{ }\right\rangle, \ldots,\left\langle a_{m_{2} 2}, \sqrt{ }\right\rangle,\left\langle b_{12}, e_{12}^{\prime}\right\rangle, \ldots,\left\langle b_{n_{2} 2}, e_{n_{2} 2}^{\prime}\right\rangle\right\} \tag{A.4}
\end{align*}
$$

Then it follows from the form of the TSS rules in Definition 2.4 concerning sums of star expressions that the sets of action derivatives of $e_{1}+e_{2}$ is the union of the sets of action derivatives of $e_{1}$, and of $e_{2}$. By permuting the action derivatives with tick to the front, this union has the list representation:

$$
\begin{align*}
A \partial\left(e_{1}+e_{2}\right)= & \left\{\left\langle a_{11}, \sqrt{ }\right\rangle, \ldots,\left\langle a_{m_{1} 1}, \sqrt{ }\right\rangle,\left\langle a_{12}, \sqrt{ }\right\rangle, \ldots,\left\langle a_{m_{2} 2}, \sqrt{ }\right\rangle,\right. \\
& \left.\left\langle b_{11}, e_{11}^{\prime}\right\rangle, \ldots,\left\langle b_{n_{1} 1}, e_{n_{1} 1}^{\prime}\right\rangle,\left\langle b_{12}, e_{12}^{\prime}\right\rangle, \ldots,\left\langle b_{n_{2} 2}, e_{n_{2} 2}^{\prime}\right\rangle\right\} \tag{A.5}
\end{align*}
$$

Now we can argue as follows to reassemble $e_{1}+e_{2}$ from its action derivatives:

$$
e \equiv e_{1}+e_{2}=\mathrm{BBP}\left(\left(\sum_{i=1}^{m_{1}} a_{i 1}\right)+\left(\sum_{j=1}^{n_{1}} b_{j 1} \cdot e_{j 1}^{\prime}\right)\right)+\left(\left(\sum_{i=1}^{m_{2}} a_{i 2}\right)+\left(\sum_{j=1}^{n_{2}} b_{j 2} \cdot e_{j 2}^{\prime}\right)\right)
$$

(by the induction hypothesis, using representation (A.5))

$$
=\mathrm{BBP}\left(\left(\left(\sum_{i=1}^{m_{1}} a_{i 1}\right)+\left(\sum_{i=1}^{m_{2}} a_{i 2}\right)\right)+\left(\sum_{j=1}^{n_{1}} b_{j 1} \cdot e_{j 1}^{\prime}\right)\right)+\left(\sum_{j=1}^{n_{2}} b_{j 2} \cdot e_{j 2}^{\prime}\right) .
$$

(by axioms (B2) and (B1))
Since ACl is a subsystem of BBP , this chain of provably equalities is one in BBP. It demonstrates, together with applications of the axiom (B2) that are needed to bring each of the subexpressions of the two outermost summands into a form with association of summation subterms to the left, that $e$ satisfies (A.1) when we construe $A \partial(e)$ in (A.5) as a list representation of the form (A.2) with $m=m_{1}+m_{2}$ and $n=n_{1}+n_{2}$.
Case 4: $e \equiv e_{1} \cdot e_{2}$.
As argued in the previous case, we may assume that the action derivatives of $e_{1}$ are of the form:

$$
\begin{equation*}
A \partial\left(e_{1}\right)=\left\{\left\langle a_{11}, \sqrt{ }\right\rangle, \ldots,\left\langle a_{m_{1} 1}, \sqrt{ }\right\rangle,\left\langle b_{11}, e_{11}^{\prime}\right\rangle, \ldots,\left\langle b_{n_{1} 1}, e_{n_{1} 1}^{\prime}\right\rangle\right\} \tag{A.6}
\end{equation*}
$$

Then it follows from the forms of the two rules in the TSS in Definition 2.4 concerning transitions from expressions with concatenation as their outermost symbol that the set of action derivatives of $e_{1} \cdot e_{2}$ has a list representation of the form:

$$
\begin{equation*}
A \partial\left(e_{1} \cdot e_{2}\right)=\left\{\left\langle a_{11}, e_{2}\right\rangle, \ldots,\left\langle a_{m_{1} 1}, e_{2}\right\rangle,\left\langle b_{11}, e_{11}^{\prime} \cdot e_{2}\right\rangle, \ldots,\left\langle b_{n_{1} 1}, e_{n_{1} 1}^{\prime} \cdot e_{2}\right\rangle\right\} \tag{A.7}
\end{equation*}
$$

Case 4.1: $m_{1}, n_{1}>0$.
Then we can reassemble $e_{1} \cdot e_{2}$ as follows:

$$
\begin{aligned}
e \equiv e_{1} \cdot e_{2} & ={ }_{\mathrm{BBP}}\left(\left(\sum_{i=1}^{m_{1}} a_{i 1}\right)+\left(\sum_{j=1}^{n_{1}} b_{j 1} \cdot e_{j 1}^{\prime}\right)\right) \cdot e_{2} \\
& ={ }_{\mathrm{BBP}}\left(\sum_{i=1}^{m_{1}} a_{i 1} \cdot e_{2}\right)+\left(\sum_{j=1}^{n_{1}}\left(b_{j 1} \cdot e_{j 1}^{\prime}\right) \cdot e_{2}\right) \\
& ={ }_{\mathrm{BBP}}\left(\sum_{i=1}^{m_{1}} a_{i 1} \cdot e_{2}\right)+\left(\sum_{j=1}^{n_{1}} b_{j 1} \cdot\left(e_{j 1}^{\prime} \cdot e_{2}\right)\right) \\
& ={ }_{\mathrm{BBP}} 0+\left(\left(\sum_{i=1}^{m_{1}} a_{i 1} \cdot e_{2}\right)+\left(\sum_{j=1}^{n_{1}} b_{j 1} \cdot\left(e_{j 1}^{\prime} \cdot e_{2}\right)\right)\right)
\end{aligned}
$$

(by the induction hypothesis, using representation (A.6))
(by axiom (B4))
(by axiom (B5))
(by axiom (B6))

This chain of provable equalities demonstrates, together with applications of the axiom (B2) that are needed to bring each of the subexpressions of the right outermost summands into a form with association of summation subterms to the left, that $e$ satisfies (A.1) when we construe $A \partial(e)$ in (A.7) as a list representation (A.2) with $m=0$ and $n=m_{1}+n_{1}$.

Case 4.2: $m_{1}>0, n_{1}=0$.
Then we can reassemble $e_{1} \cdot e_{2}$ as follows:

$$
\begin{array}{rlr}
e \equiv e_{1} \cdot e_{2} & ={ }_{\text {BBP }}\left(\left(\sum_{i=1}^{m_{1}} a_{i 1}\right)+\left(\sum_{j=1}^{n_{1}} b_{j 1} \cdot e_{j 1}^{\prime}\right)\right) \cdot e_{2} & \begin{array}{c}
\text { (by the induction hypothesis, } \\
\text { using representation (A.6)) }
\end{array} \\
& ={ }_{\text {BBP }}\left(\left(\sum_{i=1}^{m_{1}} a_{i 1}\right)+0\right) \cdot e_{2} & \text { (since } \left.\left.n_{1}=0\right)\right) \\
& ={ }_{\text {BBP }}\left(\sum_{i=1}^{m_{1}} a_{i 1} \cdot e_{2}\right)+0 \cdot e_{2} & \text { (by axiom B4) } \\
& ={ }_{\text {BBP }}\left(\sum_{i=1}^{m_{1}} a_{i 1} \cdot e_{2}\right)+0 & \text { (by axiom (B7)) } \tag{byaxiomB4}
\end{array}
$$

$$
={ }_{\mathrm{BBP}} 0+\left(\sum_{i=1}^{m_{1}} a_{i 1} \cdot e_{2}\right)
$$

(by axioms (B1) and (B6))
This chain of provable equalities demonstrates that $e$ satisfies (A.1) when we construe $A \partial(e)$ in (A.7), recalling that $n_{1}=0$, as a list representation (A.2) with $m=0$ and $n=m_{1}$.
Case 4.3: $m_{1}=0, n_{1}>0$.
Then we can reassemble $e_{1} \cdot e_{2}$ as follows:

$$
\begin{array}{rlr}
e \equiv e_{1} \cdot e_{2} & ={ }_{\mathrm{BBP}}\left(\left(\sum_{i=1}^{m_{1}} a_{i 1}\right)+\left(\sum_{j=1}^{n_{1}} b_{j 1} \cdot e_{j 1}^{\prime}\right)\right) \cdot e_{2} & \begin{array}{c}
\text { (by the induction hypothesis, } \\
\text { using representation (A.6)) }
\end{array} \\
& ={ }_{\mathrm{BBP}}\left(0+\left(\sum_{j=1}^{n_{1}} b_{j 1} \cdot e_{j 1}^{\prime}\right)\right) \cdot e_{2} & \\
& ={ }_{\mathrm{BBP}} 0 \cdot e_{2}+\left(\sum_{j=1}^{n_{1}}\left(b_{j 1} \cdot e_{j 1}^{\prime}\right) \cdot e_{2}\right) & \\
& ={ }_{\text {BBP }} 0+\left(\sum_{j=1}^{n_{1}}\left(b_{j 1} \cdot e_{j 1}^{\prime}\right) \cdot e_{2}\right) & \text { (by axiom (B4)) } \\
& ={ }_{\text {BBP }} 0+\left(\sum_{j=1}^{n_{1}} b_{j 1} \cdot\left(e_{j 1}^{\prime} \cdot e_{2}\right)\right) & \text { (by axiom (B7)) } \\
& &
\end{array}
$$

This chain of provable equalities demonstrates that $e$ satisfies (A.1) when we construe $A \partial(e)$ in (A.7), recalling that $m_{1}=0$, as a list representation (A.2) with $m=0$ and $n=n_{1}$.
Case 4.4: $m_{1}=n_{1}=0$.
Then we can reassemble $e_{1} \cdot e_{2}$ as follows:

$$
\begin{array}{rlr}
e \equiv e_{1} \cdot e_{2} & ={ }_{\text {BBP }}\left(\left(\sum_{i=1}^{m_{1}} a_{i 1}\right)+\left(\sum_{j=1}^{n_{1}} b_{j 1} \cdot e_{j 1}^{\prime}\right)\right) \cdot e_{2} & \begin{array}{c}
\text { (by the induction hypothesis, } \\
\text { using representation }(\mathrm{A} .6))
\end{array} \\
& ={ }_{\text {BBP }}(0+0) \cdot e_{2} & \left(\text { since } m_{1}=n_{1}=0\right) \\
& ={ }_{\text {BBP }} 0 \cdot e_{2} & (\text { by axiom }(\mathrm{B} 6)) \\
& ={ }_{\text {BBP }} 0 & \text { (by axiom (B7)) } \\
& ={ }_{\text {BBP }} 0+0 & \text { (by axiom (B6)) }
\end{array}
$$

This chain of provable equalities demonstrates that $e$ satisfies (A.1) when we construe $A \partial(e)$ in (A.7), recalling that $m_{1}=n_{1}=0$, as a list representation (A.2) with $m=0$ and $n=0$.

Case 5: $e \equiv e_{1}{ }^{\circledast} e_{2}$.
As in Case 3 we may assume that the sets of action derivatives of the constituent expressions $e_{1}$ and $e_{2}$ of $e_{1}+e_{2}$ have list representations of the form (A.4). Then it follows from the forms of the three rules in Definition 2.4 concerning transitions from expressions with binary iteration as their outermost symbol, that the set of action derivatives of $e_{1} \overbrace{2}$ has a list representation of the form:

$$
\begin{aligned}
A \partial\left(e_{1} e_{2}\right)= & \left\{\left\langle a_{11}, e_{1} \circledast e_{2}\right\rangle, \ldots,\left\langle a_{m_{1} 1}, e_{1}{ }^{\circledast} e_{2}\right\rangle,\right. \\
& \left\langle b_{11}, e_{11}^{\prime} \cdot\left(e_{1}{ }^{\circledast} e_{2}\right)\right\rangle, \ldots,\left\langle b_{n_{1} 1}, e_{n_{1} 1}^{\prime} \cdot\left(e_{1} e_{e_{2}}\right)\right\rangle, \\
& \left.\left\langle a_{12}, \sqrt{ }\right\rangle, \ldots,\left\langle a_{m_{2} 2}, \sqrt{ }\right\rangle,\left\langle b_{12}, e_{12}^{\prime}\right\rangle, \ldots,\left\langle b_{n_{2} 2}, e_{n_{2} 2}^{\prime}\right\rangle\right\} .
\end{aligned}
$$

By permuting the action derivatives with tick to the front, this representation can be changed into:

$$
\begin{align*}
A \partial\left(e_{1} e_{2}\right)= & \left\{\left\langle a_{12}, \sqrt{ }\right\rangle, \ldots,\left\langle a_{m_{2} 2}, \sqrt{ }\right\rangle,\right. \\
& \left\langle a_{11}, e_{1}^{\circledast} e_{2}\right\rangle, \ldots,\left\langle a_{m_{1} 1}, e_{1} \circledast e_{2}\right\rangle,  \tag{A.8}\\
& \left\langle b_{11}, e_{11}^{\prime} \cdot\left(e_{1} \circledast e_{2}\right)\right\rangle, \ldots,\left\langle b_{n_{1} 1}, e_{n_{1} 1}^{\prime} \cdot\left(e_{1} e_{2}\right)\right\rangle, \\
& \left.\left\langle b_{12}, e_{12}^{\prime}\right\rangle, \ldots,\left\langle b_{n_{2} 2}, e_{n_{2} 2}^{\prime}\right\rangle\right\} .
\end{align*}
$$

Now we argue as follows in order to reassemble $e_{1} e_{2}$ from its action derivatives in $A \partial(e)$ :

$$
\begin{aligned}
& e \equiv \quad e_{1}{ }^{\circledast} e_{2} \\
& ={ }_{\text {BBP }} e_{1} \cdot\left(e_{1}{ }^{\circledast} e_{2}\right)+e_{2} \\
& \left.={ }_{\mathrm{BBP}}\left(\sum_{i=1}^{m_{1}} a_{i 1}\right)+\left(\sum_{j=1}^{n_{1}} b_{j 1} \cdot e_{j 1}^{\prime}\right) \cdot\left(e_{1} \circledast_{e_{2}}\right)\right)+\left(\left(\sum_{i=1}^{m_{2}} a_{i 2}\right)+\left(\sum_{j=1}^{n_{2}} b_{j 2} \cdot e_{j 2}^{\prime}\right)\right) \\
& ={ }_{\mathrm{BBP}}\left(\left(\sum_{i=1}^{m_{1}} a_{i 1} \cdot\left(e_{1} \circledast_{2}\right)\right)+\left(\sum_{j=1}^{n_{1}}\left(b_{j 1} \cdot e_{j 1}^{\prime}\right) \cdot\left(e_{1} \circledast_{2}\right)\right)\right)+\left(\left(\sum_{i=1}^{m_{2}} a_{i 2}\right)+\left(\sum_{j=1}^{n_{2}} b_{j 2} \cdot e_{j 2}^{\prime}\right)\right) \\
& \text { (by axiom (B4)) } \\
& ={ }_{\mathrm{BBP}}\left(\left(\sum_{i=1}^{m_{1}} a_{i 1} \cdot\left(e_{1} \circledast_{2}\right)\right)+\left(\sum_{j=1}^{n_{1}} b_{j 1} \cdot\left(e_{j 1}^{\prime} \cdot\left(e_{1} \circledast_{2}\right)\right)\right)\right)+\left(\left(\sum_{i=1}^{m_{2}} a_{i 2}\right)+\left(\sum_{j=1}^{n_{2}} b_{j 2} \cdot e_{j 2}^{\prime}\right)\right) \\
& \text { (by axiom (B5)) } \\
& =\mathrm{ACI}\left(\sum_{i=1}^{m_{2}} a_{i 2}\right)+\left(\left(\sum_{i=1}^{m_{1}} a_{i 1} \cdot\left(e_{1} \circledast e_{2}\right)\right)+\left(\left(\sum_{j=1}^{n_{1}} b_{j 1} \cdot\left(e_{j 1}^{\prime} \cdot\left(e_{1} e_{2}\right)\right)\right)+\left(\sum_{j=1}^{n_{2}} b_{j 2} \cdot e_{j 2}^{\prime}\right)\right)\right) \\
& \text { (assumption in this case) } \\
& \text { (by axiom (BKS1)) } \\
& \text { (by the induction hypothesis, } \\
& \text { using representation (A.4)) }
\end{aligned}
$$

This chain of provably equalities demonstrates, together with applications of the axiom (B2) that are needed to bring each of the subexpressions of the right outermost summand into a form with association of summation subterms to the left, that $e$ satisfies (A.1) when we construe $A \partial(e)$ in (A.8) as a list representation of the form (A.2) with $m=m_{2}$ and $n=m_{1}+n_{1}+n_{2}$.
In each of these five possible cases concerning the outermost structure of $e$ we have successfully performed the induction step. In this way we have proved the statement of the lemma.

Proof of Proposition 2.9. Let $C(e)=\langle V(e), \sqrt{ }, e, A, T(e)\rangle$ be the chart interpretation of a star expression $e \in \operatorname{StExp}(A)$.
Let $f \in V(e) \subseteq \operatorname{StExp}(A)$ be a vertex of $C(e)$. By Lemma A. 2 every star expression in $\operatorname{StExp}(A)$ can be reassembled as the BBP-provable sum over products of over its action derivatives $\langle a, \xi\rangle$, that is, over all actions $a \in A$ and $a$-derivatives $\xi$ of $e$. In particular, (A.1) guarantees that $\mathrm{id}_{V(e)}(f)=f$ satisfies the condition for $\mathrm{id}_{V(e)}$ to be a provable solution at the vertex $f$ of $C(e)$, relative to a representation (A.2) of the action derivatives of $f$ which corresponds to a representation as assumed in Definition 2.8. Since $f \in V(e)$ was arbitrary in this argument, it follows that $\mathrm{id}_{V(e)}$ is a provable solution of $\mathcal{C}(e)$.

## A. 2 Proofs in Section 3: Layered loop existence and elimination

Proposition (= Proposition 3.7). For every $e \in \operatorname{StExp}(A)$, the entry/body-labeling $\widehat{C(e)}$ of $C(e)$ is a LLEE-witness of C(e).
Proof. To verify (W1) it suffices to show that there are no infinite body step paths from any star expression $e$ (this is also a preparation for (W2)(a), part (L2)). We prove, by induction on the syntactic structure of $e$, the stronger statement that if $e \rightarrow^{+} f$, then there does not exist an infinite body step path from $f$. The base cases, in which $e$ is of the form $a$ or 0 , are trivial. Suppose $e \equiv e_{1}+e_{2}$. Then $e_{i} \rightarrow^{+} f$ for some $i \in\{1,2\}$. So by induction, $f$ does not exhibit an infinite body step path. Suppose $e \equiv e_{1} \cdot e_{2}$. Then $e \rightarrow^{+} f$ means either $e_{1} \rightarrow^{+} f_{1}$ and $f \equiv f_{1} \cdot e_{2}$, or $e_{2} \rightarrow^{*} f$. In the first case, by induction, $f_{1}$ and $e_{2}$ do not exhibit infinite body step paths. This induces that $f_{1} \cdot e_{2}$ does not exhibit an infinite body step path. In the second case, by induction, $f$ does not exhibit an infinite body step path. Suppose $e \equiv e_{1}{ }^{\circledast} e_{2}$. Then $e^{+} f$ means (A) $f \equiv e_{1}{ }^{\circledast} e_{2}$, or (B) $e_{1} \rightarrow^{+} f_{1}$ and $f \equiv f_{1} \cdot\left(e_{1}{ }^{\circledast} e_{2}\right)$, or (C) $e_{2} \rightarrow^{+} f$. In case (A), each body step path from $f$ starts with either $f \rightarrow$ bo $e_{1}^{\prime} \cdot\left(e_{1}{ }^{\circledast} e_{2}\right)$ where $e_{1} \rightarrow e_{1}^{\prime}$ and $e_{1}^{\prime}$ is not normed, or $f \rightarrow$ bo $e_{2}^{\prime}$ where $e_{2} \rightarrow e_{2}^{\prime}$. In the first case, by induction, $e_{1}^{\prime}$ does not exhibit an infinite body step path, so since $e_{1}^{\prime}$ is not normed, $e_{1}^{\prime} \cdot\left(e_{1}{ }^{\circledast} e_{2}\right)$ does not exhibit an infinite body step path. In the second case, by induction, $e_{2}^{\prime}$ does not exhibit an infinite body step path. In case (B), since by induction $f_{1}$ and by case (A) $e_{1}{ }^{\circledast} e_{2}$ do not exhibit infinite body step paths, $f_{1} \cdot\left(e_{1} e_{2}\right)$ does not exhibit an infinite body step path. In case (C), by induction, $f$ does not exhibit an infinite body step path.

We verify (W2). From the TSS-rules in Definition 2.4 it follows that if $e$ has a loop-entry transition, then $e \equiv\left((\cdots)\left(e_{1}{ }^{\circledast} e_{2}\right) \cdot\right.$ $\left.\left.f_{1}\right) \cdots\right) \cdot f_{n}$ ) for some $n \geqslant 0$ and $e_{1}$ normed. Let $\hat{C}$ denote the entry/body-labeling defined by the TSS-rules in Definition 2.4 on the 'free' (= start-vertex free) chart of all star expressions in $\operatorname{StExp}(A)$. We prove (W2) for a subchart $C_{\hat{C}}(e, \alpha)$ of $\hat{\mathcal{C}}$. We first consider the case $n=0$, and then generalize it.

Let $e \equiv e_{1} \circledast e_{2}$ with $e_{1}$ normed, and $\alpha=\left|e_{1}\right| \circledast+1$. Either $e \rightarrow{ }_{[\alpha]} e$ or $e \rightarrow{ }_{[\alpha]} e_{1}^{\prime} \cdot e$ for some normed $e_{1}^{\prime}$ with $e_{1} \rightarrow e_{1}^{\prime}$. In the first case (L1) is clearly satisfied; we focus on the second case. It can be argued, by induction on syntactic structure, that every normed star expression has a body step path to $\sqrt{ }$. Then so does $e_{1}^{\prime}$. This means $e_{1}^{\prime} \cdot e$ has a body step path to $e$. Hence (L1) holds. For the remainder of (W2) it suffices to consider loop-entry transitions $e \rightarrow[\alpha] e_{1}^{\prime \prime} \cdot e$ where $e_{1} \rightarrow e_{1}^{\prime \prime}$. Since we showed above there are no body step cycles, every body step path from $e_{1}^{\prime \prime}$ eventually leads to deadlock or $\sqrt{ }$; in the first case the corresponding body step path of $e_{1}^{\prime \prime} \cdot e$ also deadlocks, and in the second case it returns to $e$. Hence (L2) holds. Since $e_{1}^{\prime \prime} \cdot e$ cannot reach $\sqrt{ }$ without returning to $e$, (L3) holds. It can be shown, by induction on derivation depth, that $f \rightarrow f^{\prime}$ implies $|f|_{\circledast} \geqslant\left|f^{\prime}\right|_{\circledast}$, and clearly $f \rightarrow[\beta]$ implies $\beta \leqslant|f|_{\circledast}$. So if $e_{1}^{\prime \prime} \rightarrow^{*} . \rightarrow_{[\beta]}$, then $\beta \leqslant\left|e_{1}^{\prime \prime}\right|_{\circledast} \leqslant\left|e_{1}\right|_{\circledast}$. Hence, if $e_{1}^{\prime \prime} \cdot e \xrightarrow\left[ \pm\left(e_{1} \circledast_{\left.e_{2}\right)}\right]{ }{ }_{\text {bo }} \cdot \rightarrow[\beta] \text {, then } \beta<\left|e_{1}\right|_{\circledast}+1=\alpha \text {. So (W2)(b) holds. }\right.$

Now consider $e \equiv\left(\left(\cdots\left(\left(e_{1} e_{2}\right) \cdot f_{1}\right) \cdots\right) \cdot f_{n}\right)$ for $n>0$, with $e_{1}$ normed. Again $\alpha=\left|e_{1}\right| \circledast+1$. The subchart $C_{\hat{C}}(e, \alpha)$ basically coincides with $C_{\hat{\mathcal{C}}}\left(e_{1}{ }_{e}, \alpha\right)$, except that the star expressions in the first chart are post-fixed with $f_{1}, \ldots, f_{n}$; its transitions are derived by $n$ additional applications of the first rule for concatenation in Definition 2.4, to affix these expressions. This chart isomorphism between $\mathcal{C}_{\hat{\mathcal{C}}}\left(e_{1}{ }^{\circledast} e_{2}, \alpha\right)$ and $C_{\hat{\mathcal{C}}}(e, \alpha)$ preserves action labels as well as the loop-labeling, because the first rule for concatenation preserves these labels. We showed that $C_{\hat{C}}\left(e_{1}{ }^{\circledast} e_{2}, \alpha\right)$ satisfies (W2), so the same holds for $C_{\hat{C}}(e, \alpha)$.

We now turn to the proof of Lemma 3.9, which expresses properties of the body transition relation $\rightarrow_{\mathrm{bo}}$, the descends-in-loop-to relation $\leadsto$, the loops-back-to relation $G$, and the directly-loops-back-to relation ${ }_{d} G$.
Lemma (= Lemma 3.9). The relations $\rightarrow_{\mathrm{bo}}, \triangleleft, G,{ }_{d} G$ as defined by a LLEE-witness $\hat{C}$ on a chart $C$ satisfy the following properties:
(i) $\hat{C}$ does not have infinite $\rightarrow_{\text {bo }}$ paths (so no $\rightarrow_{\text {bo }}$ cycles).
(ii) If $\operatorname{scc}(u)=\operatorname{scc}(v)$, then $u$-* $^{*}$ implies $v G^{*} u$.
(iii) If $v \frown w$ and $\neg(w)$, then $w$ is not normed.
(iv) $\operatorname{scc}(u)=\operatorname{scc}(v)$ if and only if $u G^{*} w$ and $v G^{*} w$ for some vertex $w$.
(v) $G^{*}$ is a partial order that has the least-upper-bound property: if a nonempty set of vertices has an upper bound with respect to $G^{*}$, then it has a least upper bound.
(vi) $G$ is a total order on $G$-successor vertices: if $w G v_{1}$ and $w G v_{2}$, then $v_{1} G v_{2}$ or $v_{1}=v_{2}$ or $v_{2} G v_{1}$.
(vii) If $v_{1}{ }_{d} G u$ and $v_{2} G u$ for distinct $v_{1}, v_{2}$, then there is no vertex $w$ such that both $w G^{*} v_{1}$ and $w G^{*} v_{2}$.

We split the proof into the arguments for the parts (i)-(vii), respectively. In doing so we repeat these statements as individual lemmas, and add a few more on the way.

Proof. By induction on the number $n$ of $\leftrightharpoons$-steps in a path $\left.v^{\alpha}\right\lrcorner \cdot \frown^{n} \cdot \rightarrow[\beta]$. If $n=0$, then from $v^{\alpha} \frown \cdot \rightarrow[\beta]$ we get $\alpha>\beta$ by means of the LLEE-witness condition (W2)(b). If $n>0$, then the path $v^{\alpha} \frown \cdot ゝ^{n} \cdot \rightarrow_{[\beta]}$ is of the form $v^{\alpha} \frown \cdot \frown^{n-1} \cdot \gamma \frown \cdot \rightarrow[\beta]$ for some loop name $\gamma$. This path contains an initial segment $v^{\alpha} \frown \cdot \frown^{n-1} \cdot \rightarrow[\gamma]$. Then $\alpha>\gamma$ follows by the induction hypothesis. From the part $\gamma \triangleleft \cdot \rightarrow_{[\beta]}$ of this path we get $\gamma>\beta$ by LLEE-witness condition (W2)(b). So we conclude that $\alpha>\beta$ holds.

Lemma A.4. In a chart with a LLEE-witness, if $v \frown^{+} w$, then $w \neq \sqrt{ }$.
Proof. Let $\hat{C}$ be a LLEE-witness of a chart $\mathcal{C}$. It suffices to show that $\frown w$ implies $w \neq \sqrt{ }$. For this, we let $v$ and $w$ be vertices such that $v \frown w$. Then we can pick $\alpha \in \mathbb{N}^{+}$such that $v^{\alpha} \frown w$. Since this means $v \underset{\mathbf{f ( v )}}{[\alpha]} \cdot \underset{\mathbf{t ( v )}}{ }{ }_{\text {bo }}^{*} w$, it follows that $w \in C_{\hat{C}}(v, \alpha)$. Now since $C_{\hat{\mathcal{C}}}(v, \alpha)$ is a loop chart by condition (W2)(a) for the LLEE-witness $\hat{\mathcal{C}}$, it follows that $w \neq \sqrt{ }$.
Lemma A.5. In a chart with a LLEE-witness (assumed to be start-vertex connected, see Definition 2.2), every vertex is reachable by an acylic $\rightarrow_{\mathrm{bo}}^{*} \cdot$ 土 $^{*}$ path from the start vertex $v_{\mathrm{s}}$, that is, $v_{\mathrm{s}} \rightarrow_{\mathrm{bo}}^{*} \cdot$ د* $^{*} w$ holds for all vertices $w$.
Proof. Let $\pi$ be a path from $v_{\text {s }}$ to $w$. By removing cycles from $\pi$ we obtain an acyclic path $\pi^{\prime}$ from $v_{\mathrm{s}}$ to $w$ that consists of a sequence of loop-entry and body transitions. Hence $\pi^{\prime}$ is of the form $v_{\mathrm{s}} \rightarrow_{\text {bo }}^{*} w$ or $v_{\mathrm{s}} \rightarrow_{\text {bo }}^{*} u_{0} \xrightarrow[\mathbf{t}(v)]{[ }\left[\alpha_{0}\right] \cdot \xrightarrow[\mathbf{t}\left(u_{0}\right)]{ }{ }_{\text {bo }}^{*}$ $u_{1} \xrightarrow[\mathbf{t}\left(u_{1}\right)]{ }\left[\alpha_{1}\right] \cdot \xrightarrow[\notin\left(u_{1}\right)]{ }{ }_{\text {bo }} \cdots \xrightarrow[\mathbf{t}\left(u_{1}\right)]{ }\left[\alpha_{1}\right] \cdot \xrightarrow[\mathfrak{t}\left(u_{n-1}\right)]{ }{ }_{\text {bo }}^{*} u_{n} \equiv w$ for some $n \in \mathbb{N}$, and $\alpha_{0}, \ldots, \alpha_{n} \in \mathbb{N}^{+}$, where the target-avoidance parts are due to acyclicity of $\pi^{\prime}$. Hence $\pi^{\prime}$ is of the form $v_{s} \rightarrow_{\text {bo }}^{*} u_{0}{ }^{\alpha_{0}} \frown . \alpha_{1} \frown \ldots \alpha_{n-2} \frown \alpha_{n-1} \frown w$, for some $n \in \mathbb{N}$, and $\alpha_{0}, \ldots, \alpha_{n} \in \mathbb{N}^{+}$, and therefore of the form $v_{\mathrm{s}} \rightarrow_{\mathrm{bo}}^{*} \cdot \sim^{*} w$.

Lemma A.6. In a chart with a LLEE-witness, for every path $v \underset{\mathbf{t ( v )}}{\longrightarrow}[\alpha] \cdot \underset{\mathbf{t ( v )}}{ }{ }^{*} w$ there is an acyclic path $v^{\alpha} \frown \cdot \frown^{*} w$.
Proof. Let $\pi$ be a path from $v$ to $w$ that starts with a loop-entry step with loop name $\alpha$ such that all targets of transitions in $\pi$ avoid $v$. By removing cycles we obtain an acyclic path $\pi^{\prime}$ from $v$ to $w$ that starts with an $\alpha$-loop-entry step whose target
 $\cdots u_{n-2} \xrightarrow[\mathbf{t}\left(u_{n-2}\right)]{ }\left[\alpha_{n-2}\right] \cdot \xrightarrow[\mathbf{t}\left(u_{n-2}\right)]{ }{ }^{*}{ }^{*} u_{n-1} \xrightarrow[\mathbf{t}\left(u_{n-1}\right)]{ }\left[\alpha_{n-1}\right] \cdot \xrightarrow[\mathbf{t}\left(u_{n-1}\right)]{ }$ bo $w$ for some $n \geqslant 1$, where the target-avoidance parts are


The following lemma was also used implicitly in the proof of Lem. 3.9, (v).
Lemma A.7. In a chart with a LLEE-witness, if $v \underset{\neq(v)}{ }[\alpha] \cdot{\underset{\mathbf{t ( v )}}{ }}^{*} \cdot \rightarrow[\beta]$, then $\alpha>\beta$.
Proof. This is a direct consequence of Lem. A. 6 and Lem. A.3.
Lemma A.8. In a chart with a LLEE-witness, if $u G^{*} v G^{*} w$, then each path $u \rightarrow_{\text {bo }}^{*} w$ visits $v$.
Proof. Let $v \neq u, w$, as else the lemma trivially holds. Since $u G^{+} v G^{+} w$, there is a path $\underset{\notin(w)}{\longrightarrow}[\alpha] \cdot \xrightarrow[\mathbf{t}(w)]{ } \quad v \underset{\mathbf{t}(v)}{ }[\beta] \cdot \longrightarrow_{\mathbf{t}(v)} * u$.
 $\beta>\alpha$, which cannot be the case.
Lemma (= Lemma 3.9, (i)). In a chart with a LLEE-witness, there are no infinite $\rightarrow_{\mathrm{bo}}$ paths (so no $\rightarrow_{\mathrm{bo}}$ cycles).
Proof. Let $C$ be a chart with LLEE-witness $\hat{C}$, and with start vertex $v_{\mathrm{s}}$. Due to Lemma A. 5 every vertex of $v$ is reachable by a $\rightarrow_{\text {bo }}^{*} \cdot \frown^{*}$ path from $v_{s}$. In order to show that there are no infinite $\rightarrow_{\text {bo }}$ paths in $\hat{C}$ it therefore suffices to show that if $v_{\mathrm{s}} \rightarrow_{\mathrm{bo}}^{*} \cdot \hookrightarrow^{n} v$, then there is no infinite $\rightarrow_{\mathrm{bo}}$ path from $v$.

For the base case, $n=0$, let $w$ be such that $v_{\text {s }} \rightarrow_{\text {bo }}^{*} w$. Now suppose that there is an infinite $\rightarrow_{\text {bo }}^{*}$ path from $w$ in $\hat{\mathcal{C}}$. Then due to $v_{\mathrm{s}} \rightarrow_{\text {bo }}^{*} w$ it follows that there is also an infinite $\rightarrow_{\text {bo }}^{*}$ path from $v_{\mathrm{s}}$ in $\hat{C}$. This, however, contradicts with the condition (W1) that the LLEE-witness $\hat{C}$ must satisfy. We conclude that there is no infinite $\rightarrow_{\text {bo }}^{*}$ path from $w$ in $\hat{\mathcal{C}}$.

For performing the induction step from $n$ to $n+1$, let $w$ be such that $v_{s} \rightarrow_{\text {bo }}^{*} \cdot \frown^{n+1} w$. Then we can pick $w_{0}$ with $v_{\mathrm{s}} \rightarrow_{\mathrm{bo}}^{*} \cdot \frown^{n} w_{0} \circlearrowright w$. It follows that $w_{0} \xrightarrow[t\left(w_{0}\right)]{ }[\alpha] \cdot \underset{t\left(w_{0}\right)}{ }{ }^{*}$ bo $w$ for some $\alpha \in \mathbb{N}^{+}$, which we pick accordingly. Now suppose that there is an infinite $\rightarrow_{\text {bo }}^{*}$ path $\pi$ from $w$ in $\hat{C}$. Then it cannot be the case that $\pi$ avoids $w_{0}$ forever, because otherwise it would give rise to an infinite path $w_{0} \xrightarrow[ \pm\left(w_{0}\right)]{ }[\alpha] \cdot \underset{\ddagger\left(w_{0}\right)}{ }$ bo $w \underset{\ddagger\left(w_{0}\right)}{ }$ bo $w_{1} \xrightarrow[ \pm\left(w_{0}\right)]{ }$ bo $w_{2} \xrightarrow[ \pm\left(w_{0}\right)]{ }$ bo $\cdots$, which is not possible since the condition (W2)(a) for the LLEE-witness $C$ implies that $C_{\hat{C}}\left(u_{0}, \alpha\right)$ is a loop chart. Therefore it follows that $\pi$ must visit $v_{0}$. But then $\pi$ also gives rise to an infinite $\rightarrow_{\text {bo }}^{*}$ path from $w_{0}$. This, however, contradicts the the statement that the induction hypothesis guarantees for $w_{0}$ due to $v_{\mathrm{s}} \rightarrow_{\text {bo }}^{*} \cdot \neg^{n} w_{0}$, namely that there is no infinite $\rightarrow_{\text {bo }}^{*}$ path from $w_{0}$. We have reached a contradiction. Therefore we can conclude that there is no infinite $\rightarrow_{\text {bo }}^{*}$ path $\pi$ from $w$ in $\hat{\mathcal{C}}$. In this way we have successfully performed the induction step.
Lemma (= Lemma 3.9, (ii)). In a chart with a LLEE-witness, if $\operatorname{scc}(u)=\operatorname{scc}(v)$, then $u \frown^{*} v$ implies $v Q^{*} u$.
Proof. We prove that $u \rightharpoonup^{n} v$ implies $v G^{n} u$ for all $n \geqslant 0$, by induction on $n$. The base case $n=0$ is trivial, as then $u=v$. If $n>0, u \frown^{n-1} u^{\prime} \frown v$ for some $u^{\prime}$. Clearly $\operatorname{scc}(u)=\operatorname{scc}\left(u^{\prime}\right)=\operatorname{scc}(v)$. By induction, $u^{\prime} G^{n-1} u$. Since $u^{\prime} \frown v$, there is an acyclic path $u^{\prime} \rightarrow[\alpha] \cdot \rightarrow_{\text {bo }}^{*} v$. And since $\operatorname{scc}\left(u^{\prime}\right)=\operatorname{scc}(v)$, there is an acyclic path $\left.v \rightarrow_{\text {bo }}^{*} \cdot \rightarrow_{\left[\beta_{1}\right]} \cdot \rightarrow_{\text {bo }}^{*} \cdots \cdots \rightarrow \beta_{k}\right] \cdot \rightarrow_{\text {bo }}^{*} u^{\prime}$. By (W2)(b), $\alpha>\beta_{1}>\cdots>\beta_{k}>\alpha$. This means $k=0$, so $v \rightarrow_{\text {bo }}^{*} u^{\prime}$. This implies $v G u^{\prime}$ and hence $v G^{n} u$.
Lemma (= Lemma 3.9, (iii)). If, in a chart with a LLEE-witness, $\neg w$ and $\neg(w G)$, then $w$ is not normed.
Proof. We argue indirectly by showing that the negation of the implication in the statement of the lemma leads to a contradiction. For this, suppose that $v \frown w$ and $\neg(w G)$ hold for some vertices $v$ and $w$, and that additionally $w$ is normed. From $v \frown w$ and $\neg(w)$ we obtain by Lem. 3.9, (ii) that $w \notin \operatorname{scc}(v)$. Since $v \frown w$ entails $v \rightarrow^{*} w$ this entails $\neg\left(w \rightarrow^{*} v\right)$. Now since that
 Then it follows from Lemma A. 6 that $v \frown^{+} \sqrt{ }$. This, however, contradicts, Lemma A.4.
Lemma (= Lemma 3.9, (iv)). In a chart with a LLEE-witness, $\operatorname{scc}(u)=\operatorname{scc}(v)$ if and only if $u G^{*} w$ and $v G^{*} w$ for some vertex $w$.

Proof. The direction from right to left of the lemma trivially holds; we focus on the direction from left to right. Let $\operatorname{scc}(u)=$ $\operatorname{scc}(v)$. The case $u=v$ is trivial. Let $u \neq v$. Then they are on a cycle, which, since there is no body step cycle, contains a loop-entry transition from some $w$. Without loss of generality, suppose $w \neq u$. Then $w \frown^{+} u$, so by Lemma 3.9, (ii), $u G^{+} w$. If $w=v$ we have $v G^{*} w$, and if $w \neq v$ we can argue in the same fashion that $v G^{+} w$.
Lemma A.9. In a chart with a LLEE-witness, $G^{+}$is irreflexive.
Proof. Let $\hat{C}$ be a LLEE-witness of a LLEE-chart $C$. Suppose that $w G^{+} w$ holds for some vertex $w$ of $C$ and $\hat{C}$. Then it follows from the definition of $G^{+}$that there is $\mathrm{a} \rightarrow_{\text {bo }}$ path of non-zero length from $w$ to $w$ itself. But such $\rightarrow_{\text {bo }}$ cycle in $\hat{C}$ is not possible, as it would give rise to an infinite $\rightarrow_{\text {bo }}$ path in $\hat{\mathcal{C}}$, contradicting Lemma 3.9, (i).

Lemma A.10. In a chart with a LLEE-witness, $G^{*}$ is a partial order.
Proof. By definition, $G$ is transitive-reflexive. Moreover, $G$ is anti-symmetric, because $u G^{+} v$ and $v G^{+} u$ for $u \neq v$ would imply $u G^{+} v$ and $v G^{+} u$, in contradiction with irreflexivity of $G^{+}$, see Lemma A.9.
Lemma (= Lemma 3.9, (v)). In a chart with a LLEE-witness, $G^{*}$ is a partial order that has the least-upper-bound property: if a nonempty set of vertices has an upper bound with respect to $G^{*}$, then it has a least upper bound.
Proof. Let $C$ be a chart with a LLEE-witness $C$. Let the relation $G$ be defined on $C$ according to $\hat{C}$.
$G^{*}$ is a partial order by Lemma A.10. Since $C$ as a chart is finite, it suffices to show that for each vertex $v$ the set of vertices $x$ with $v G^{*} x$ is totally ordered with regard to $G^{*}$. Let $v G^{+} u_{1}$ and $v G^{+} u_{2}$ with $u_{1} \neq u_{2}$. There is a path $u_{1} \xrightarrow[t\left(u_{1}\right)]{\longrightarrow}[\alpha] \cdot \xrightarrow[\neq\left(u_{1}\right)]{ }{ }^{*}$ $v \rightarrow_{\mathrm{bo}}^{+} u_{2} \underset{\ddagger\left(u_{2}\right)}{ }[\beta]{\underset{ \pm\left(u_{2}\right)}{ }}^{*} v \rightarrow_{\mathrm{bo}}^{+} u_{1}$. Without loss of generality, suppose $\beta \geqslant \alpha$. Then layeredness implies that each path $v \rightarrow_{\text {bo }}^{+} u_{2}$ must visit $u_{1}$, so $v \underset{\mathbf{t}\left(u_{2}\right)}{+}$ bo $^{+} u_{1} \rightarrow_{\text {bo }}^{+} u_{2}$. Hence there is a path $u_{2} \underset{\mathbf{t ( u _ { 2 } )}}{ }[\beta] \cdot \underset{\mathbf{t}\left(u_{2}\right)}{*} v \underset{\mathbf{t}\left(u_{2}\right)}{ }{ }_{\text {bo }}^{+} u_{1} \rightarrow_{\text {bo }}^{+} u_{2}$, which implies $u_{1} G^{+} u_{2}$.
Lemma (= Lemma 3.9, (vii)). In a chart with a LLEE-witness, if $v_{1}{ }_{d} G u$ and $v_{2}{ }_{d} G u$ for distinct $v_{1}, v_{2}$, then there is no vertex $w$ such that both $w G^{*} v_{1}$ and $w G^{*} v_{2}$.
Proof. $\neg\left(v_{2} G^{+} v_{1}\right)$ and $\neg\left(v_{1} G^{+} v_{2}\right)$, for else the definition of ${ }_{d} G$ would imply $u G^{*} v_{1}$ or $u G^{*} v_{2}$, and so $v_{1} G^{+} v_{1}$ or $v_{2} G^{+} v_{2}$, contradicting irreflexivity of $G^{+}$, see Lemma A.9. In the proof of Lemma 3.9, (v), we furthermore saw that for each $w,\left\{x \mid w G^{*} x\right\}$ is totally ordered with regard to $G^{*}$, which implies that any such sets cannot contain both $v_{1}$ and $v_{2}$.

## A. 3 Proofs in Section 5: Extraction of star expressions from, and transferral between, LLEE-charts

Proposition (= Proposition 5.1, requires BBP-axioms (B1), (B2), (B3)). Let $\phi: V_{1} \rightarrow V_{2}$ be a functional bisimulation between charts $C_{1}$ and $C_{2}$. Let $s_{2}: V_{2} \backslash\{\sqrt{ }\} \rightarrow$ StExp $(A)$ be a provable solution of $C_{2}$. Then $s_{2} \circ \phi: V_{1} \backslash\{\sqrt{ }\} \rightarrow \operatorname{StExp}(A)$ is a provable solution of $C_{1}$ with the same principal value as $s_{2}$.
Proof. Let $s_{2}$ be a provable solution of $C_{2}$. Let $v \in V_{1} \backslash\{\sqrt{ }\}$. Since $\phi$ is a functional bisimulation between $C_{1}$ and $C_{2}$, the forth, back, and termination conditions for the graph of $\phi$ as a bisimulation hold for the pair $\langle v, \phi(v)\rangle$ of vertices. This makes it possible to bring the sets of transitions $T_{1}(v)$ from $v$ in $C_{1}$, and $T_{2}(\phi(v))$ from $\phi(v)$ in $C_{2}$ into a 1-1 correspondence such that $\phi$ again relates their targets:

$$
\begin{align*}
T_{1}(v) & =\left\{v \xrightarrow{a_{i}} \sqrt{ } \mid i=1, \ldots, m\right\} \cup\left\{v \xrightarrow{b_{j}} v_{j 1}^{\prime} \mid j=1, \ldots, n\right\},  \tag{A.9}\\
T_{2}(\phi(v)) & =\left\{\phi(v) \xrightarrow{a_{i}} \sqrt{ } \mid i=1, \ldots, m\right\} \cup\left\{\phi(v) \xrightarrow{b_{j}} v_{j 2}^{\prime} \mid j=1, \ldots, n\right\},  \tag{A.10}\\
\phi\left(v_{j 1}^{\prime}\right) & =v_{j 2}^{\prime}, \text { for all } j \in\{1, \ldots, n\}, \tag{A.11}
\end{align*}
$$

with $n, m \in \mathbb{N}$, and vertices $v_{j 1}^{\prime} \in V_{1} \backslash\{\sqrt{ }\}$, and $v_{j 2}^{\prime} \in V_{2} \backslash\{\sqrt{ }\}$, for $j \in\{1, \ldots, n\}$. Note that the same transition may be listed multiple times in the set $T_{2}(\phi(v))$. On this basis we can argue as follows.

$$
\left(s_{2} \circ \phi\right)(v) \equiv s_{2}(\phi(v)) \quad={ }_{\mathrm{BBP}} \quad\left(\sum_{i=1}^{m} a_{i}\right)+\left(\sum_{j=1}^{n} b_{j} \cdot s_{2}\left(v_{j 2}^{\prime}\right)\right)
$$

(since $s_{2}$ is a provable solution of $C_{2}$, using (A.10) and axioms (B1), (B2), (B3))

$$
\begin{aligned}
& \equiv \quad\left(\sum_{i=1}^{m} a_{i}\right)+\left(\sum_{j=1}^{n} b_{j} \cdot\left(s_{2} \circ \phi\right)\left(v_{j 1}^{\prime}\right)\right) \\
& \quad\left(\text { using }(\mathrm{A} .11) \text { and }\left(s_{2} \circ \phi\right)\left(v_{j 1}^{\prime}\right) \equiv s_{2}\left(\phi\left(v_{j 1}^{\prime}\right)\right)\right) \\
& \quad 21
\end{aligned}
$$

This shows, in view of (A.9), that $s \circ \phi$ satisfies the condition for a provable solution at $v$. Now as $v \in V_{1} \backslash\{\sqrt{ }\}$ was arbitrary, $s_{2} \circ \phi$ (with domain $V_{1} \backslash\{\sqrt{ }\}$ ) is a provable solution of $C_{1}$. Since furthermore the functional bisimulation $\phi$ must relate the start vertices of $C_{1}$ and $C_{2}$, the principal value of $s_{2} \circ \phi$ coincides with that of $s_{2}$.

Lemma (= Lemma 5.2). In a chart with a LLEE-witness, for all vertices $v, w$ :
(i) $v \rightarrow_{\mathrm{bo}} w \Rightarrow\|v\|_{\mathrm{bo}}>\|w\|_{\mathrm{bo}}$,
(ii) $v \frown w \Rightarrow|v|_{\text {en }}>|w|_{\text {en }}$.

Proof. For statement (i) we argue as follows. Recall that the body step norm $\|v\|_{b o}$ in a LLEE-witness $\hat{C}$ was defined as the maximal length of a body step path from $v$ in $\hat{C}$. This was well-defined due to Lemma 3.9, (i), and the finiteness of charts. Now suppose that $v \rightarrow_{\text {bo }} w$. Then every body step path from $w$ gives rise to a body step path from $v$ that starts with the transition $v \rightarrow_{\text {bo }} w$. Hence a longest body step path from $w$ of length $\|w\|_{\text {bo }}$ gives rise to a body step path from $v$ of length $\|w\|_{\text {bo }}+1$. It follows that $\|v\|_{\text {bo }} \geqslant\|w\|_{\text {bo }}+1>\|w\|_{\text {bo }}$, and hence $\|v\|_{\text {bo }}>\|w\|_{\text {bo }}$.

For showing statement (ii), suppose that $v \frown w$. Then $v{ }^{\alpha} \frown w$ holds for some $\alpha \in \mathbb{N}^{+}$. Then $|v|_{\text {en }} \geqslant \alpha$. If there is no loop-entry transition that departs from $w$, then $|w|_{\text {en }}=0$ holds, and hence we get $|v|_{\text {en }} \geqslant \alpha>0=|w|_{\text {en }}$. Otherwise we let $\beta \in \mathbb{N}^{+}$be the maximal index of a loop-entry transition from $w$. Then $v^{\alpha} \frown w \rightarrow[\beta]$. By Lemma A. 3 it follows that $\alpha>\beta$. Consequently we find $|v|_{\mathrm{en}} \geqslant \alpha>\beta=|w|_{\mathrm{en}}$. In both cases we have shown $|v|_{\mathrm{en}}>|w|_{\mathrm{en}}$.

Lemma (= Lemma 5.4, uses the BBP-axioms (B1)-(B6), (BKS2), but not the rule RSP ${ }^{\circledast}$ ). For a LLEE-chart $C$ with LLEE-witness $\hat{C}$ the following connection holds between the extracted solution $s_{\hat{C}}$ and the relative extracted solution $t_{\hat{C}}$, for all vertices $v, w$ :

$$
\begin{equation*}
v \frown w \Longrightarrow s_{\hat{C}}(w)=\text { BBP } t_{\hat{C}}(w, v) \cdot s_{\hat{C}}(v) \tag{A.12}
\end{equation*}
$$

Note that if $v \frown w$, then $v \neq \sqrt{ }$, and also $w \neq \sqrt{ }$, because $w$ is in the body of a loop at $v$, and therefore cannot be $\sqrt{ }$ (see Lem. A.4).
Proof. In order to show (A.12) we proceed by complete induction (without explicit treatment of the base case) on the length $\|w\|_{\text {bo }}$ of a longest body step path from $w$. For performing the induction step, we consider arbitrary $v, w \neq \sqrt{ }$ with $v \frown w$. We assume a representation of the set $\hat{T}(w)$ of transitions from $w$ in $\hat{C}$ :

$$
\begin{align*}
\hat{T}(w)= & \left\{w \xrightarrow{a_{i}}\left[\alpha_{i}\right] w \mid i=1, \ldots, m\right\} \cup\left\{w \xrightarrow{b_{j}}\left[\beta_{j}\right] w_{j} \mid w_{j} \neq w, j=1, \ldots, n\right\} \\
& \cup\left\{w \xrightarrow{c_{i}} \text { bo } v \mid i=1, \ldots, p\right\} \cup\left\{w \xrightarrow{d_{j}} \text { bo } u_{j} \mid u_{j} \neq v, j=1, \ldots, q\right\} \tag{A.13}
\end{align*}
$$

that partitions $\hat{T}(w)$ into loop-entry transitions to $w$ and to other targets $w_{1}, \ldots, w_{n}$, and body transitions to $v$ and to other targets $u_{1}, \ldots, u_{q}$. Since $w$ is contained in a loop at $v$, none of these targets can be $\sqrt{ }$. In order to show provable equality at the right-hand side of (A.12), we argue as follows:

$$
s_{\hat{\mathcal{C}}}(w) \equiv\left(\left(\sum_{i=1}^{m} a_{i}\right)+\left(\sum_{j=1}^{n} b_{j} \cdot t_{\hat{\mathcal{C}}}\left(w_{j}, w\right)\right)\right)^{\circledast}\left(0+\left(\left(\sum_{i=1}^{p} c_{i} \cdot s_{\hat{\mathcal{C}}}(v)\right)+\left(\sum_{j=1}^{q} d_{j} \cdot s_{\hat{\mathcal{C}}}\left(u_{j}\right)\right)\right)\right)
$$

(by the definition of $s_{\hat{C}}(w)$, based on the representation (A.13), using that none of the target vertices is $\sqrt{ }$ )

$$
\begin{aligned}
& ={ }_{\mathrm{BBP}}\left(\left(\sum_{i=1}^{m} a_{i}\right)+\left(\sum_{j=1}^{n} b_{j} \cdot t_{\hat{\mathrm{C}}}\left(w_{j}, w\right)\right)\right)^{\circledast}\left(\left(\sum_{i=1}^{p} c_{i} \cdot s_{\hat{C}}(v)\right)+\left(\sum_{j=1}^{q} d_{j} \cdot s_{\hat{C}}\left(u_{j}\right)\right)\right) \\
& \quad \quad \text { using axiom (B6)) } \\
& ={ }_{\mathrm{BBP}}\left(\left(\sum_{i=1}^{m} a_{i}\right)+\left(\sum_{j=1}^{n} b_{j} \cdot t_{\hat{C}}\left(w_{j}, w\right)\right)\right)^{\circledast}\left(\left(\sum_{i=1}^{p} c_{i} \cdot s_{\hat{C}}(v)\right)+\left(\sum_{j=1}^{q} d_{j} \cdot\left(t_{\hat{C}}\left(u_{j}, v\right) \cdot s_{\hat{C}}(v)\right)\right)\right)
\end{aligned}
$$

(by the induction hypothesis, using that $v \frown u_{j}$ and $\left\|u_{j}\right\|_{\text {bo }}<\|w\|_{\text {bo }}$ because $w \rightarrow_{\text {bo }} u_{j}$ for $j=1, \ldots, q$, see (A.13))

$$
={ }_{\mathrm{BBP}}\left(\left(\sum_{i=1}^{m} a_{i}\right)+\left(\sum_{j=1}^{n} b_{j} \cdot t_{\hat{C}}\left(w_{j}, w\right)\right)\right)^{\circledast}\left(\left(\left(\sum_{i=1}^{p} c_{i}\right)+\left(\sum_{j=1}^{q} d_{j} \cdot t_{\hat{C}}\left(u_{j}, v\right)\right)\right) \cdot s_{\hat{C}}(v)\right)
$$

(using axioms (B5), (B4))

$$
\begin{aligned}
& =\text { BBP }^{\left(\left(\sum_{i=1}^{m} a_{i}\right)+\left(\sum_{j=1}^{n} b_{j} \cdot t_{\hat{\mathcal{C}}}\left(w_{j}, w\right)\right)\right)^{\circledast}\left(\left(\sum_{i=1}^{p} c_{i}\right)+\left(\sum_{j=1}^{q} d_{j} \cdot t_{\hat{\mathcal{C}}}\left(u_{j}, v\right)\right)\right) \cdot s_{\hat{\mathcal{C}}}(v)} \\
& \quad \quad \text { (using axiom (BKS2)) } \\
& \equiv \quad t_{\hat{\mathcal{C}}}(w, v) \cdot s_{\hat{\mathcal{C}}}(v) \\
& \quad \text { (by the definition of } t_{\hat{\mathcal{C}}}(w, v) \text {, based on the representation (A.13)) }
\end{aligned}
$$

This chain of provable equalities demonstrates (A.12).
Proposition (= Proposition 5.5, uses the BBP-axioms (B1)-(B6), (BKS1), (BKS2), but not the rule RSP ${ }^{\circledast}$ ). In a chart $C$ with a LLEE-witness $\hat{\mathcal{C}}, s_{\hat{C}}$ is a provable solution of $C$.
Proof. We prove that $s_{\hat{C}}$ is a provable solution of the chart $C$. Let $w \neq \sqrt{ }$. We show that $s_{\hat{C}}(w)$ satisfies the defining equation of $s_{\hat{C}}$ to be a provable solution of $C$ at $w$.

We consider a representation of the set $\hat{T}(w)$ of transitions from $w$ in $\hat{C}$ as follows:

$$
\begin{align*}
\hat{T}(w)= & \left\{w \xrightarrow{a_{i}}\left[\alpha_{i}\right] w \mid i=1, \ldots, m\right\} \cup\left\{w \xrightarrow{b_{j}}\left[\beta_{j}\right] w_{j} \mid w_{j} \neq w, j=1, \ldots, n\right\} \\
& \cup\left\{w \xrightarrow{c_{i}} \text { bo } \sqrt{ } \mid i=1, \ldots, p\right\} \cup\left\{w \xrightarrow{d_{j}} \text { bo } u_{j} \mid u_{j} \neq \sqrt{ }, j=1, \ldots, q\right\} \tag{A.14}
\end{align*}
$$

that partitions $\hat{T}(w)$ into loop-entry transitions to $w$ and to other targets $w_{1}, \ldots, w_{n}$, and body transitions to $\sqrt{ }$ and to other targets $u_{1}, \ldots, u_{q}$. We argue as follows:

$$
\begin{aligned}
& s_{\hat{\mathcal{C}}}(w) \equiv\left(\left(\sum_{i=1}^{m} a_{i}\right)+\left(\sum_{j=1}^{n} b_{j} \cdot t_{\hat{\mathcal{C}}}\left(w_{j}, w\right)\right)\right)^{\circledast}\left(\left(\sum_{i=1}^{p} c_{i}\right)+\left(\sum_{j=1}^{q} d_{j} \cdot s_{\hat{C}}\left(u_{j}\right)\right)\right) \\
& \text { (by the definition of } s_{\hat{C}} \text {, in view of (A.14)) } \\
& ={ }_{\mathrm{BBP}}\left(\left(\sum_{i=1}^{m} a_{i}\right)+\left(\sum_{j=1}^{n} b_{j} \cdot t_{\hat{\mathcal{C}}}\left(w_{j}, w\right)\right)\right) \cdot s_{\hat{\mathcal{C}}}(w)+\left(\left(\sum_{i=1}^{p} c_{i}\right)+\left(\sum_{j=1}^{q} d_{j} \cdot s_{\hat{\mathcal{C}}}\left(u_{j}\right)\right)\right) \\
& \text { (using axiom (BKS1) and the defining equality in the first step) } \\
& ={ }_{\mathrm{BBP}}\left(\left(\sum_{i=1}^{m} a_{i} \cdot s_{\hat{C}}(w)\right)+\left(\sum_{j=1}^{n} b_{j} \cdot\left(t_{\hat{\mathrm{C}}}\left(w_{j}, w\right) \cdot s_{\hat{C}}(w)\right)\right)\right)+\left(\left(\sum_{i=1}^{p} c_{i}\right)+\left(\sum_{j=1}^{q} d_{j} \cdot s_{\hat{C}}\left(u_{j}\right)\right)\right) \\
& \text { (using axioms (B5), (B4)) } \\
& ={ }_{\mathrm{BBP}}\left(\left(\sum_{i=1}^{m} a_{i} \cdot s_{\hat{C}}(w)\right)+\left(\sum_{j=1}^{n} b_{j} \cdot s_{\hat{\mathcal{C}}}\left(w_{j}\right)\right)\right)+\left(\left(\sum_{i=1}^{p} c_{i}\right)+\left(\sum_{j=1}^{q} d_{j} \cdot s_{\hat{C}}\left(u_{j}\right)\right)\right) \\
& \text { (using (A.12) of Lemma 5.4, in view of } w \frown w_{j} \text { for } j=1, \ldots, n \text { ) } \\
& ={ }_{\mathrm{BBP}}\left(\sum_{i=1}^{p} c_{i}\right)+\left(\left(\left(\sum_{i=1}^{m} a_{i} \cdot s_{\hat{C}}(w)\right)+\left(\sum_{j=1}^{n} b_{j} \cdot s_{\hat{\mathcal{C}}}\left(w_{j}\right)\right)\right)+\left(\sum_{j=1}^{q} d_{j} \cdot s_{\hat{C}}\left(u_{j}\right)\right)\right)
\end{aligned}
$$

(using axioms (B1), (B2))
This chain of provable equalities demonstrates that $s_{\hat{C}}(w)$ is a provable solution of $C$ at $w$, in view of (A.14). As $w \neq \sqrt{ }$ is arbitrary, $s_{\hat{C}}$ is indeed a provable solution of $C$.

Lemma (= Lemma 5.7, uses the BBP-axioms (B1)-(B6), and the rule RSP ${ }^{\circledast}$ ). For every provable solution s of a chart $C$ with LLEE-witness $\hat{\mathcal{C}}$, the following connection holds with the relative extraction function $t_{\hat{C}}$ holds, for all vertices $v, w$ :

$$
\begin{equation*}
v \frown w \Longrightarrow s(w)=\mathrm{BBP} t_{\hat{C}}(w, v) \cdot s(v) \tag{A.15}
\end{equation*}
$$

Note that if $v \frown w$, then $v \neq \sqrt{ }$, and also $w \neq \sqrt{ }$, because $w$ is in the body of a loop at $v$, and therefore cannot be $\sqrt{ }$.
Proof. In order to prove (A.15) we proceed by complete induction on the same measure as used in the definition of the relative extraction function $t_{\hat{C}}$, namely, induction on the maximal loop level of a loop at $v$, with a subinduction on $\|w\|_{\text {bo }}$. For performing the induction step, consider vertices $v, w$ with $v \frown w$. As in the proof of Prop. 5.5 we assume the representation (A.13) of the set $\hat{T}(w)$ of transitions from $w$ in $\hat{C}$, which partitions $\hat{T}(w)$ into loop-entry transitions to $w$ and to other targets $w_{1}, \ldots, w_{n}$,
and body transitions to $v$ and to other targets $u_{1}, \ldots, u_{q}$. Since $w$ is contained in a loop at $v$, none of these targets can be $\sqrt{ }$. We now argue as follows:

$$
s(w)={ }_{\mathrm{BBP}} 0+\left(\left(\sum_{i=1}^{m} a_{i} \cdot s(w)\right)+\left(\left(\sum_{j=1}^{n} b_{j} \cdot s\left(w_{j}\right)\right)+\left(\sum_{i=1}^{p} c_{i} \cdot s(v)\right)+\left(\sum_{j=1}^{q} d_{j} \cdot s\left(u_{j}\right)\right)\right)\right)
$$

(since $s$ is a provable solution of $C$ at $w$, using (A.13))

$$
\begin{aligned}
& =\operatorname{BBP}\left(\left(\sum_{i=1}^{m} a_{i} \cdot s(w)\right)+\left(\sum_{j=1}^{n} b_{j} \cdot s\left(w_{j}\right)\right)\right)+\left(\left(\sum_{i=1}^{p} c_{i} \cdot s(v)\right)+\left(\sum_{j=1}^{q} d_{j} \cdot s\left(u_{j}\right)\right)\right) \\
& \quad \quad \quad \text { using axioms (B6), (B2)) } \\
& =\operatorname{BBP}\left(\left(\sum_{i=1}^{m} a_{i} \cdot s(w)\right)+\left(\sum_{j=1}^{n} b_{j} \cdot\left(t_{\hat{\mathcal{C}}}\left(w_{j}, w\right) \cdot s(w)\right)\right)\right)+\left(\left(\sum_{i=1}^{p} c_{i} \cdot s(v)\right)+\left(\sum_{j=1}^{q} d_{j} \cdot\left(t_{\hat{\mathcal{C}}}\left(u_{j}, v\right) \cdot s(v)\right)\right)\right)
\end{aligned}
$$

(using the induction hypothesis, which is applicable because the maximal loop level at $w$ is smaller than that at $v$ due to $v \frown w$, and
$v \frown u_{i}$ and $\left\|u_{j}\right\|_{\text {bo }}<\|w\|_{\text {bo }}$ due to $w \rightarrow_{\text {bo }} u_{j}$ for $j=1, \ldots, q$, see (A.13))
$={ }_{\mathrm{BBP}}\left(\left(\sum_{i=1}^{m} a_{i}\right)+\left(\sum_{j=1}^{n} b_{j} \cdot t_{\hat{\mathcal{C}}}\left(w_{j}, w\right)\right)\right) \cdot s(w)+\left(\left(\sum_{i=1}^{p} c_{i}\right)+\left(\sum_{j=1}^{q} d_{j} \cdot t_{\hat{\mathcal{C}}}\left(u_{j}, v\right)\right)\right) \cdot s(v)$
(using axioms (B5), (B4))
This chain of provable equalities justifies:

$$
s(w)=\operatorname{BBP}\left(\left(\sum_{i=1}^{m} a_{i}\right)+\left(\sum_{j=1}^{n} b_{j} \cdot t_{\hat{C}}\left(w_{j}, w\right)\right)\right) \cdot s(w)+\left(\left(\sum_{i=1}^{p} c_{i}\right)+\left(\sum_{j=1}^{q} d_{j} \cdot t_{\hat{C}}\left(u_{j}, v\right)\right)\right) \cdot s(v)
$$

To this equality we can apply the rule RSP ${ }^{\circledast}$ :

$$
\left.\begin{array}{rl}
s(w)= & { }_{\mathrm{BBP}}\left(\left(\left(\sum_{i=1}^{m} a_{i}\right)+\left(\sum_{j=1}^{n} b_{i} \cdot t_{\hat{C}}\left(w_{j}, w\right)\right)\right)^{\circledast}\left(\left(\sum_{i=1}^{p} c_{i}\right)+\left(\sum_{j=1}^{q} d_{j} \cdot t_{\hat{C}}\left(u_{j}, v\right)\right)\right)\right) \cdot s(v) \\
& \quad(\text { by applying rule RSP }
\end{array}{ }^{\circledast}\right) .
$$

The last step uses the definition of $t_{\hat{C}}(w, v)$, based on representation (A.13) of $\widehat{T}(w)$. In this way we have carried out the induction step. We conclude that (A.15) holds for all vertices $v$ and $w$ of $C$.

## A. 4 Proofs in Section 6: Preservation of LLEE under collapse

Lemma (= Lemma 6.2). If $w_{1} \leftrightarrows w_{2}$ in $C$, then $C_{w_{2}}^{\left(w_{1}\right)} \leftrightarrows C$.
Proof. Let $C=\left\langle V_{1}, \sqrt{ }, v_{\mathrm{s}, 1}, T_{1}\right\rangle$ and $C_{w_{2}}^{\left(w_{1}\right)}=\left\langle V_{2}, \sqrt{ }, v_{\mathrm{s}, 2}, T_{2}\right\rangle$. Let $B_{1} \subseteq V_{1} \times V_{1}$ be the largest bisimulation relation on $C$. In particular, $\left\langle w_{1}, w_{2}\right\rangle \in B_{1}$. We argue that $B_{2}=B_{1} \cap\left(V_{1} \times V_{2}\right)$ is a bisimulation relation between $C$ and $C_{w_{2}}^{\left(w_{1}\right)}$. Take any $\langle u, v\rangle \in B_{2} \subseteq B_{1}$.

- (forth): Let $u \xrightarrow{a} u^{\prime} \in T_{1}$. Then $\langle u, v\rangle \in B_{1}$ implies there is a $v \xrightarrow{a} v^{\prime} \in T_{1}$ with $\left\langle u^{\prime}, v^{\prime}\right\rangle \in B_{1}$. If $v \xrightarrow{a} v^{\prime} \in T_{2}$, then $v^{\prime} \in V_{2}$, so $\left\langle u^{\prime}, v^{\prime}\right\rangle \in B_{2}$ and we are done. If $v \xrightarrow{a} v^{\prime} \notin T_{2}$, then $v^{\prime}=w_{1}$ and $v \xrightarrow{a} w_{2} \in T_{2}$. Since $\left\langle u^{\prime}, w_{1}\right\rangle \in B_{1}$ and $\left\langle w_{1}, w_{2}\right\rangle \in B_{1}$, also $\left\langle u^{\prime}, w_{2}\right\rangle \in B_{1}$. Since $w_{2} \in V_{2}$, it follows that $\left\langle u^{\prime}, w_{2}\right\rangle \in B_{2}$.
- (back): Let $v \xrightarrow{a} v^{\prime} \in T_{2}$. If $v \xrightarrow{a} v^{\prime} \in T_{1}$, then $\langle u, v\rangle \in B_{1}$ implies there is a $u \xrightarrow{a} u^{\prime} \in T_{1}$ with $\left\langle u^{\prime}, v^{\prime}\right\rangle \in B_{1}$. Since $v^{\prime} \in V_{2}$, also $\left\langle u^{\prime}, v^{\prime}\right\rangle \in B_{2}$ and we are done. If $v \xrightarrow{a} v^{\prime} \notin T_{1}$, then $v^{\prime}=w_{2}$ and $v \xrightarrow{a} w_{1} \in T_{1}$. So $\langle u, v\rangle \in B_{1}$ implies there is a $u \xrightarrow{a} u^{\prime} \in T_{1}$ with $\left\langle u^{\prime}, w_{1}\right\rangle \in B_{1}$. Since $\left\langle u^{\prime}, w_{1}\right\rangle \in B_{1}$ and $\left\langle w_{1}, w_{2}\right\rangle \in B_{1}$, also $\left\langle u^{\prime}, w_{2}\right\rangle \in B_{1}$. Since $w_{2} \in V_{2}$, it follows that $\left\langle u^{\prime}, w_{2}\right\rangle \in B_{2}$.
- (termination): Since $B_{2} \subseteq B_{1}$ clearly $u=\sqrt{ }$ if and only if $v=\sqrt{ }$.

Finally, concerning (start): If $v_{\mathrm{s}, 1}=v_{\mathrm{s}, 2}$, then trivially $\left\langle v_{\mathrm{s}, 1}, v_{\mathrm{s}, 2}\right\rangle \in B_{2}$. If $v_{\mathrm{s}, 1} \neq v_{\mathrm{s}, 2}$, then $v_{\mathrm{s}, 1}=w_{1}$ and $v_{\mathrm{s}, 2}=w_{2}$. Since $\left\langle w_{1}, w_{2}\right\rangle \in B_{1}$ and $w_{2} \in V_{2}$, we have $\left\langle w_{1}, w_{2}\right\rangle \in B_{2}$.

Proposition (= Proposition 6.4). If a LLEE-chart $C$ is not a bisimulation collapse, then it contains a pair of bisimilar vertices $w_{1}, w_{2}$ that satisfy, for a LLEE-witness of $C$, one of the following conditions:
(C1) $\neg\left(w_{2} \rightarrow^{*} w_{1}\right) \wedge\left(\neg w_{1} \Longrightarrow w_{2}\right.$ is not normed $)$,
(C2) $w_{2} G^{+} w_{1}$,
(C3) $\exists v \in V\left(w_{1}{ }_{d} G v \wedge w_{2} G^{+} v\right) \wedge \neg\left(w_{2} \rightarrow_{\text {bo }}^{*} w_{1}\right)$.
More supplementary illustrations for the proof of Prop. 6.4 on pages $9-10$. The proof started from a pair $u_{1}, u_{2}$ of distinct bisimilar vertices. In the case $\operatorname{scc}\left(u_{1}\right)=\operatorname{scc}\left(u_{2}\right)$, we had the following situation:

$$
\begin{equation*}
u_{1} G^{*} v_{1}{ }_{d} G v_{d} \bigcirc v_{2} \bigcirc^{*} u_{2} \wedge \neg\left(v_{2} \rightarrow_{\text {bo }}^{*} v_{1}\right) \tag{A.16}
\end{equation*}
$$

For pairs of vertices $u_{1}$ and $u_{2}$ such that (A.16) holds, for some $v_{1}, v_{2}$, and $v$, we used induction on $\left\|u_{1}\right\|_{l b}^{\min }$ in order show that $u_{1}$ and $u_{2}$ progress, via pairs of distinct bisimilar vertices, to bisimilar vertices $w_{1}$ and $w_{2}$ such that one of the conditions (C1), (C2) , or (C3) holds. Note that each of (C1), (C2) , and (C3) implies that $w_{1}$ and $w_{2}$ are distinct.

In order to carry out the induction step we used a case distinction. Below we repeat the arguments, and supplement them with illustrations.
Case 1: $u_{2} G^{+} v_{2}$.
Since $u_{2} \rightarrow u_{2}^{\prime}$, either $u_{2}^{\prime}=v_{2}$ or $v_{2} \frown^{+} u_{2}^{\prime}$. Moreover, $\operatorname{scc}\left(u_{2}^{\prime}\right)=\operatorname{scc}\left(u_{2}\right)=\operatorname{scc}\left(v_{2}\right)$, so by Lem. 3.9, (ii), $u_{2}^{\prime}$ G $^{*} v_{2}$. Hence, $u_{1}^{\prime} G^{*} v_{1}{ }_{d} G v_{d} \bigcirc v_{2} \wp^{*} u_{2}^{\prime} \wedge \neg\left(v_{2} \rightarrow_{\text {bo }}^{*} v_{1}\right)$, and $\left\|u_{1}^{\prime}\right\|_{l b}^{\min }<\left\|u_{1}\right\|_{l b}^{\min }$. We apply the induction hypothesis to obtain a bisimilar pair $w_{1}, w_{2}$ for which (C1), (C2) , or (C3) holds. In the illustration below, we drew both of the two cases in which the transition $u_{2} \rightarrow u_{2}^{\prime}$ is a loop-entry transition, or a body transition, from $u_{2}$.


Case 2: $u_{2}=v_{2}$.

## Case 2.1: $u_{2} \rightarrow{ }_{[\alpha]} u_{2}^{\prime}$.

Then either $u_{2}^{\prime}=u_{2}$ or $u_{2} \frown u_{2}^{\prime}$. Moreover, $\operatorname{scc}\left(u_{2}^{\prime}\right)=\operatorname{scc}\left(u_{2}\right)$, so by Lem. 3.9, (ii), $u_{2}^{\prime} G^{*} u_{2}$, and hence $u_{2}^{\prime} G^{*} v_{2}$. Thus we have obtained $u_{1}^{\prime} G^{*} v_{1}{ }_{d} G v_{d} \bigcirc v_{2} \bigcirc^{*} u_{2}^{\prime} \wedge \neg\left(v_{2} \rightarrow_{\text {bo }}^{*} v_{1}\right)$. Due to $\left\|u_{1}^{\prime}\right\|_{l b}^{\min }<\left\|u_{1}\right\|_{l b}^{\min }$, we can apply the induction hypothesis again.


Case 2.2: $u_{2} \rightarrow$ bo $u_{2}^{\prime}$.
Then $\neg\left(v_{2} \rightarrow_{\text {bo }}^{*} v_{1}\right)$ together with $v_{2}=u_{2} \rightarrow_{\text {bo }} u_{2}^{\prime}$ and $u_{1}^{\prime} \rightarrow_{\text {bo }}^{*} v_{1}$ (because $\left.u_{1}^{\prime} G^{*} v_{1}\right)$ imply $u_{1}^{\prime} \neq u_{2}^{\prime}$. We distinguish two cases.
Case 2.2.1: $u_{2}^{\prime}=v$.
Then $u_{1}^{\prime} G^{*} v_{1}{ }_{d} G v=u_{2}^{\prime}$, i.e., $u_{1}^{\prime} G^{+} u_{2}^{\prime}$, so we are done, because (C2) holds for $w_{1}=u_{2}^{\prime}$ and $w_{2}=u_{1}^{\prime}$.


Case 2.2.2: $u_{2}^{\prime} \neq v$.
By Lem. 3.9, (ii), $u_{2}^{\prime} G^{+} v$. Hence, $u_{2}^{\prime} G^{*} v_{2}^{\prime}{ }_{d} G v$ for some $v_{2}^{\prime}$. Since $v_{2}=u_{2} \rightarrow_{\text {bo }} u_{2}^{\prime} G^{*} v_{2}^{\prime}$ and $\neg\left(v_{2} \rightarrow_{\text {bo }}^{*} v_{1}\right)$, it follows that $\neg\left(v_{2}^{\prime} \rightarrow_{\text {bo }}^{*} v_{1}\right)$. So $u_{1}^{\prime} G^{*} v_{1}{ }_{d} G v_{d} \bigcirc v_{2}^{\prime} \bigcirc^{*} u_{2}^{\prime} \wedge \neg\left(v_{2} \rightarrow_{\text {bo }}^{*} v_{1}^{\prime}\right)$. Due to $\left\|u_{1}^{\prime}\right\|_{l b}^{\min }<\left\|u_{1}\right\|_{l b}^{\min }$, we can apply the induction hypothesis again.


Proposition (= Proposition 6.8). Let $C$ be a LLEE-chart. If a pair $\left\langle w_{1}, w_{2}\right\rangle$ of vertices satisfies (C1), (C2), or (C3) with respect to a LLEE-witness of $C$, then $C_{w_{2}}^{\left(w_{1}\right)}$ is a LLEE-chart.
As background for the proof of this proposition, we first give examples why conditions (C1), (C2), and (C3) cannot be readily relaxed or changed. These examples showcase that, far from being artificial, the conditions (C1), (C2), and (C3) mark sharp borders between whether, on a given LLEE-witness, a connect-through operation is possible while preserving LLEE, or not. Thus these examples demonstrate that a further simplification of the case analysis provided by Proposition 7.3 is not readily possible, with an eye towards LLEE-structure preserving connect-through operations. Therefore a substantial further improvement of our stepwise collapse procedure appears unlikely.

For convenience, the pictures in these examples neglect action labels on transitions.
Example A. 11 (= Example 6.3). To show that in (C1) it is crucial that $w_{1}$ does not loop back, we refer back to the LLEE-witness $\hat{C}$ in Ex. 6.3. There $\neg\left(w_{2} \rightarrow^{*} w_{1}\right)$, but (C1) is not satisfied by the pair $w_{1}, w_{2}$ because $w_{1} G \widehat{w}_{1}$. Since in $\hat{C}$ the levels of loop-entry transitions that descend to $w_{1}$ are higher than the loop levels that descend from $w_{2}$, the preprocessing step of transformation I is void. We observed that the connect- $w_{1}$-through-to- $w_{2} \operatorname{chart} C_{w_{2}}^{\left(w_{1}\right)}$ on the left in Ex. 6.3 has no LLEE-witness. The bisimilar pair $w_{1}, w_{2}$ in $\hat{C}$ progresses to the bisimilar pair $\widehat{w}_{1}, \widehat{w}_{2}$, for which (C1) holds. Since $\hat{C}_{\widehat{w}_{2}}^{\left(\widehat{w}_{1}\right)}$ on the right of Ex. 6.3
is obtained by applying transformation I to this pair, it is guaranteed to be a LEE-witness; this will be argued in the proof of Prop. 6.8.


To avoid the creation of body step cycles in transformation II, it would seem expedient to connect transitions to $w_{2}$ through to $w_{1}$, since (C2), $w_{2} G^{+} w_{1}$, rules out the existence of a path $w_{1} \rightarrow_{\text {bo }}^{+} w_{2}$ in $\hat{C}$. (Instead, transitions to $w_{1}$ are connected through to $w_{2}$, and resulting body step cycles are eliminated by turning the body transitions at $\widehat{w}_{2}$ into loop-entry transitions.) However, connecting transitions to $w_{2}$ through to $w_{1}$ may produce a chart for which no LLEE-witness exists.

Example A. 12 (= Example 6.6). For the LLEE-chart $C$ with LLEE-witness $\hat{C}$ below in the middle, the connect- $w_{2}$-through-to- $w_{1}$ chart $C_{w_{1}}^{\left(w_{2}\right)}$ on the left does not have a LLEE-witness: it has no loop subchart, because from each of its three vertices an infinite path starts that does not return to this vertex. From $\widehat{w}_{2}$ this path, drawn in red, cycles between $u$ and $w_{1}$. Transformation II applied to the pair $w_{1}, w_{2}$ (instead of $w_{2}, w_{1}$ ) in $\hat{C}$ yields the entry/body-labeling $\hat{C}_{w_{2}}^{\left(w_{1}\right)}$ for the connect- $w_{1}$-through-to- $w_{2}$ chart with additionally $\widehat{w}_{2} \rightarrow_{\text {bo }} w_{2}$ turned into $\widehat{w}_{2} \rightarrow[2] w_{2}$. Since the pair $w_{1}, w_{2}$ satisfies (C2), the proof of Prop. 6.8 guarantees that this entry/body-labeling, drawn on the right, is a LLEE-witness.


The following example shows that for transformation III it is essential to select a bisimilar pair $w_{1}, w_{2}$ where $w_{1}$ directly loops back to $v$.

Example A. 13 (= Example 6.7). In the LLEE-witness $\hat{C}$ below in the middle, $w_{1}, w_{2} G^{+} v$, and there is no body step path from $w_{2}$ to $w_{1}$, but (C3) does not hold for the pair $w_{1}, w_{2}$ because $\neg\left(w_{1}{ }_{d} G v\right)$. All loop-entry transitions from $v$ have the same loop label, so the preprocessing step of transformation III is void. The connect-w $w_{1}$-through-to- $w_{2}$ chart $C_{w_{2}}^{\left(w_{1}\right)}$ on the left does not have a LLEE-witness. Namely, the transition from $\widehat{w}_{2}$ can be declared a loop-entry transition, and after its removal also two transitions from $v$ can be declared loop-entry transitions, leading to the removal of the five transitions that are depicted as dotted arrows. The remaining chart (of solid arrows) however has no further loop subchart, because from each of its vertices an infinite path starts that does not return to this vertex. The bisimilar pair $w_{1}, w_{2}$ progresses to the bisimilar pair $\widehat{w}_{1}, \widehat{w}_{2}$ in $\hat{C}$, for which (C3) holds because $\widehat{w}_{1}{ }_{d} G v \supset \widehat{w}_{2}$ and $\neg\left(\widehat{w}_{2} \rightarrow_{\text {bo }}^{*} \widehat{w}_{1}\right)$. Transformation III applied to this pair yields the entry/body-labeling $\hat{C}_{\widehat{w}_{2}}^{\left(\widehat{w}_{1}\right)}$ on the right. In the proof of Prop. 6.8 it is argued that this is guaranteed to be a LLEE-witness. The remaining two bisimilar pairs can be eliminated by one or two further applications of transformation III.


The following example shows (C3) cannot be weakened by dropping $\neg\left(w_{2} \rightarrow_{\text {bo }}^{*} w_{1}\right)$.
Example A.14. For the LLEE-witness $\hat{C}$ below in the middle, $w_{1}{ }_{d} G v \frown^{+} w_{2}$, but there is a body step path from $w_{2}$ to $w_{1}$. The connect- $w_{1}$-through-to- $w_{2} \operatorname{chart} C_{w_{2}}^{\left(w_{1}\right)}$ on the left does not have a LLEE-witness, because from each of its vertices an infinite path starts that does not return to it. The bisimilar pair $w_{1}, w_{2}$ in $\hat{C}$ progresses to the bisimilar pair $v, \widehat{w}_{2}$, to which transformation II is applicable because (C2) holds: $\widehat{w}_{2} G v$. In the resulting LLEE-witness $\hat{C}_{\widehat{w}_{2}}^{(v)}$, second to the right, (C3) holds for the pair $w_{1}, w_{2}$ because $w_{1}{ }_{d} G \widehat{w}_{2} \bigcirc w_{2}$ and $\neg\left(w_{2} \rightarrow_{\text {bo }}^{*} w_{1}\right)$. Applying transformation III to this pair results in the LLEE-witness on the right.


Supplement for the proof of Proposition 6.8. Let $\hat{C}$ be a LLEE-chart. For vertices $w_{1}, w_{2}$ such that (C1), (C2), or (C3) holds, transformation I, II, or III, respectively, produces an entry/body-labeling $\hat{\mathcal{C}}_{w_{2}}^{\left(w_{1}\right)}$. In the article submission we have proved for transformation I that it is a LLEE-witness. Here we do the same for transformations II and III.

We recall that in the proof in the article submission we have shown that it suffices to show that each of the transformations produces, before the final clean-up step, an entry/body-labeling that satisfies the LLEE-conditions with the exception of possible violations of the loop property (L1) in (W2)(a).
Transformation II: We argue the correctness of transformation II. Consider vertices $w_{1}, w_{2}$ such that (C2) holds, that is, $w_{2} G^{+} w_{1}$. Let $\widehat{w}_{2}$ be the ${ }_{d} G$-predecessor of $w_{1}$ in the ${ }_{d} G$-chain from $w_{2}$ to $w_{1}$, i.e., $w_{2} G^{*} \widehat{w}_{2}{ }_{d} G w_{1}$.
As for the transformations I and III it suffices to show, in view of the alleviation of the proof obligation at the start of the proof on page 12, that the intermediate result $\hat{C}^{\prime \prime}$ of transformation II before the clean-up step satisfies the LLEE-witness properties, except for possible violations of (L1). By the definition of transformation II, $\hat{C}^{\prime \prime}$ results by performing the adaptation step $L_{\text {II }}$ to the chart $\hat{\mathcal{C}}^{\prime}:=\hat{\mathcal{C}}_{w_{2}}^{\left(w_{1}\right)}$ that arises from $\hat{C}$ by connecting $w_{1}$ through to $w_{2}$.
To prove that (W1), and the part concerning (L2) for (W2)(a) is satisfied for $\hat{C}^{\prime \prime}$, it suffices to show that the transformed chart does not contain a cycle of body transitions. At first, the step of connecting $w_{1}$ through to $w_{2}$ in $\hat{C}$ may introduce a body step cycle in $\hat{C}^{\prime}=\hat{\mathcal{C}}_{w_{2}}^{\left(w_{1}\right)}$. But every such cycle is removed in the subsequent level adaptation step $L_{\text {II }}$. Namely, each body step cycle introduced in $\hat{C}^{\prime}$ must stem from a transition $u \rightarrow_{\text {bo }} w_{1}$ (which is redirected to $w_{2}$ in $\hat{C}^{\prime}$ ) and a path $w_{2} \rightarrow_{\text {bo }}^{*} u$ in $\hat{C}$, for some $u \neq w_{1}$. Since $w_{2} G^{*} \widehat{w}_{2}{ }_{d} G w_{1}$, by Lem. A.8, the path $w_{2} \rightarrow_{\text {bo }}^{*} u \rightarrow$ bo $w_{1}$ in $\hat{C}$ must visit $\widehat{w}_{2}$. Since all body transitions from $\widehat{w}_{2}$ are turned into loop-entry transitions in step $L_{\mathrm{II}}$, the body step cycle $w_{2} \rightarrow_{\text {bo }}^{*} u \rightarrow_{\text {bo }} w_{2}$ in $\hat{C}^{\prime}$ that was introduced in the connect-through step, is after step $L_{\text {II }}$ no longer a body step cycle in $\hat{C}^{\prime \prime}$.

Now we prove that (W2)(b) is preserved by the two steps from $\hat{C}$ via $C^{\prime}=\hat{C}_{w_{2}}^{\left(w_{1}\right)}$ to $\hat{C}^{\prime \prime}$. Every path $u \underset{\neq(u)}{[\alpha]} \cdot{ }_{\mathbf{t}(u)}^{\longrightarrow}$ bo $\cdot \rightarrow[\beta]$ in $\hat{C}^{\prime \prime}$ with $u \neq w_{1}, w_{2}$ arises by a, possibly empty, combination of the following three kinds of modifications in the first two transformation steps:
(i) A transition to $w_{1}$ was redirected to $w_{2}$ in the connect-through step.
(ii) The loop-entry transition at the beginning of the path is from $\widehat{w}_{2}$ and was a body transition before step $L_{\text {II }}$, meaning that $u=\widehat{w}_{2}$ and $\alpha=\gamma$. (Recall that $\gamma$ is a loop name of maximum loop level among the loop-entries at $w_{1}$ in $\hat{C}$.)
(iii) The loop-entry transition at the end of the path is from $\widehat{w}_{2}$ and was a body transition before step $L_{\mathrm{II}}$, meaning that $\beta=\gamma$.
This gives $2^{3}=8$ possibilities. Of these, three possibilities are void: if all three adaptations are not the case, the path is already present in $\hat{\mathcal{C}}$, and so $\alpha>\beta$ is guaranteed; (ii) and (iii) together cannot hold, because then the path would return to $u=\widehat{w}_{2}$, which it cannot, because all of its steps avoid $u$ as target. We now show that in the remaining five cases always $\alpha>\beta$. Since $w_{2} G^{+} w_{1}$, there is a path $w_{1} \xrightarrow[\mathbf{t ( w _ { 1 } )}]{ }[\delta] \cdot \xrightarrow[\mathbf{t ( w _ { 1 } )}]{ }{ }^{*}{ }^{\text {bo }} w_{2}$ in $\hat{C}$. By definition of $\gamma, \gamma \geqslant \delta$.
 is a path $u \underset{\mathbf{t}(u)}{ }[\alpha] \cdot \xrightarrow[\mathbf{t}(u)]{ }{ }_{\text {bo }}^{*} w_{1} \rightarrow[\gamma]$ in $\hat{C}$, so $\alpha>\gamma$. We distinguish two cases.
CASE 1: The path $w_{2} \xrightarrow[\mathbf{t}(u)]{\text { bo }}{ }^{*} \rightarrow[\beta]$ visits $w_{1}$. Then there is a path $w_{1} \xrightarrow[\mathbf{t}(u)]{ }^{*} \cdot \rightarrow[\beta]$ in $\hat{C}$. So $u \underset{\mathbf{t}(u)}{\longrightarrow}[\alpha] \cdot \xrightarrow[\mathfrak{t}(u)]{ }$ * $w_{1} \xrightarrow[(u)]{ }{ }^{\text {bo }} \cdot \rightarrow{ }_{[\beta]}$ in $\hat{C}$. So by (W2)(b), $\alpha>\beta$.
CASE 2: The path $w_{2} \underset{t(u)}{ }{ }_{\text {bo }} \cdot \rightarrow[\beta]$ does not visit $w_{1}$. Then there is a path $w_{1} \xrightarrow[t\left(w_{1}\right)]{[\delta]} \cdot \underset{t\left(w_{1}\right)}{ }{ }_{\text {bo }}^{*} w_{2} \xrightarrow[t\left(w_{1}\right)]{*}{ }_{\text {bo }}^{*} \cdot \rightarrow[\beta]$ in $\hat{C}$, so $\delta>\beta$. Hence $\alpha>\gamma \geqslant \delta>\beta$.
B Let only (ii) hold. Then $u=\widehat{w}_{2}, \alpha=\gamma$, and there is a path $\widehat{w}_{2} \xrightarrow[t\left(\hat{w}_{2}, w_{1}\right)]{ }{ }_{\text {bo }}^{+} \rightarrow[\beta]$ in $\hat{C}$. As $\widehat{w}_{2}{ }_{d}{ }^{( } w_{1}$, there is a path $w_{1} \xrightarrow[ \pm\left(w_{1}\right)]{ }[\delta] \cdot \xrightarrow[ \pm\left(w_{1}\right)]{ }{ }_{\text {bo }}^{*} \widehat{w}_{2}$ in $\hat{C}$. Hence $w_{1} \xrightarrow[ \pm\left(w_{1}\right)]{ }[\delta] \cdot \underset{ \pm\left(w_{1}\right)}{ }{ }^{*}$ bo $\widehat{w}_{2} \xrightarrow[ \pm\left(w_{1}\right)]{\text { bo }}+\rightarrow_{[\beta]}^{+}$in $\hat{C}$, so $\delta>\beta$. Hence $\alpha=\gamma \geqslant \delta>\beta$.

C Let only (iii) hold. Then $\beta=\gamma$, and $u \xrightarrow[\ddagger\left(u, w_{1}\right)]{ }[\alpha] \cdot \xrightarrow[\ddagger\left(u, w_{1}\right)]{ }{ }^{*}$ bo $\widehat{w}_{2}$ with $u \neq w_{1}$ is a path in in $\hat{C}$. Since $\widehat{w}_{2} G w_{1}$ and $u \neq w_{1}$, it follows that $\neg\left(\hat{w}_{2} G \mathcal{G}\right)$. So in view of the path $u \underset{+(u)}{\longrightarrow}[\alpha] \cdot \underset{ \pm(u)}{\longrightarrow}{ }_{\text {bo }}^{*} \widehat{w}_{2}$, there is no path $\hat{w}_{2} \rightarrow_{\text {bo }}^{*} u$ in $\hat{C}$. Since $\widehat{w}_{2}{ }_{d} G w_{1}$, there is a path $\widehat{w}_{2} \rightarrow_{\text {bo }}^{*} w_{1}$ in $\hat{C}$, which by the previous observation is of the form $\widehat{w}_{2} \xrightarrow[\mathbf{t}(u)]{ } \quad$ bo $w_{1}$. Hence there is a path $u \underset{\mathbf{t}(u)}{ }[\alpha] \cdot \xrightarrow[\mathbf{t}(u)]{ }{ }_{\text {bo }}^{*} \widehat{w}_{2} \xrightarrow[\mathbf{t}(u)]{ }_{\text {bo }}^{*} w_{1} \rightarrow_{[\gamma]}$ in $\hat{C}$, so $\alpha>\gamma=\beta$.
D Let only (i) and (ii) hold, meaning $u=\widehat{w}_{2}, \alpha=\gamma$, and there are paths $\widehat{w}_{2} \underset{\boldsymbol{t ( \hat { w } _ { 2 } )}}{+} w_{1}$ and $w_{2} \xrightarrow[\boldsymbol{f ( \hat { w } _ { 2 } )}]{ }{ }^{*} \cdot \rightarrow[\beta]$ in $\hat{C}$. Since $w_{2} G^{*} \widehat{w}_{2}{ }_{d} G^{+} w_{1}$, and $u=\widehat{w}_{2}$ implies $w_{2} \neq \widehat{w}_{2}$, by Lem. A.8, the path $w_{2} \underset{\neq\left(\hat{w}_{2}\right)}{ }{ }^{*} \cdot \rightarrow{ }_{[\beta]}$ cannot visit $w_{1}$. Hence $w_{1} \xrightarrow[\mathbf{t}\left(w_{1}\right)]{ }[\delta] \cdot \xrightarrow[\mathbf{t ( w _ { 1 } )}]{ }{ }_{\text {bo }} w_{2} \xrightarrow[\boldsymbol{t}_{\left(w_{1}\right)}]{ }{ }_{\text {bo }}^{*} \cdot \rightarrow[\beta]$ in $\hat{C}$. So $\delta>\beta$. Hence $\alpha=\gamma \geqslant \delta>\beta$.
 $u \underset{\mathbf{t}(u)}{ }[\alpha] \cdot \xrightarrow[\mathbf{t}(u)]{ }{ }^{*}$ oo $w_{1} \rightarrow[\gamma]$ in $\hat{C}, \alpha>\gamma=\beta$.
We conclude that in all five cases, $\hat{C}^{\prime \prime}$ satisfies (W2)(b).
Finally we argue that part (L3) of (W2)(a) holds for $\hat{C}^{\prime \prime}$, i.e., there are no descends-in-loop-to paths of the form $u \underset{\neq(u)}{\longrightarrow}[\alpha] \cdot \xrightarrow[\neq(u)]{ }$ bo $\sqrt{*}$ in $\hat{C}^{\prime \prime}$. We can use part of the argumentation employed for demonstrating (W2)(b) above. It was demonstrated in particular that for every descends-in-loop-to path $u \underset{\mathbf{t ( u )}}{\longrightarrow}[\alpha] \cdot \xrightarrow[\mathbf{t}(u)]{ } \quad \underset{\text { bo }}{*} x$ in $\hat{C}^{\prime \prime}$, there is a descends-in-loop-to path $\tilde{u} \underset{\mathbf{t ( \tilde { u } )}}{ }[\gamma] \cdot \xrightarrow{\ddagger(\vec{u})}{ }^{*}$ bo $x$ with the same target $x$ in $\hat{C}$. From this it follows that if a descends-in-loop-to path in $\hat{C}^{\prime \prime}$ had $\sqrt{ }$ as target, then there were a descends-in-loop-to path already in $\hat{C}$ that had $\sqrt{ }$ as target, violating (L3) for the LLEE-chart $\hat{C}$. Hence $\hat{C}^{\prime \prime}$ must satisfy (L3).
We conclude that the result of transformation II is a LLEE-chart.

Transformation III: To show the correctness of transformation III, consider vertices $w_{1}$ and $w_{2}$ such that (C3) holds. Let $v$ be such that $w_{1}{ }_{d} G v \wp^{+} w_{2}$. We show that its intermediate result $\hat{C}_{w_{2}}^{\left(w_{1}\right)}$ before the clean-up step satisfies the LLEE-witness properties, except for possible violations of (L1).
First we show that (W2)(b) is preserved by both the level adaptation and the connect-through step. A violation arising by the first step, i.e., in $\hat{C}^{\prime}$, would involve a path $u \underset{ \pm(u)}{ }[\alpha] \cdot{ }_{t(u)}{ }^{*}$ bo $v \rightarrow[\beta]$ in $\hat{C}$ where $\beta$ is increased to a loop label $\gamma$ of maximum level among all loop-entries at $v$. But in this way no violation can arise, since there was already a path $u \underset{\mathbf{t}(u)}{\longrightarrow}[\alpha] \cdot \underset{\mathbf{t}(u)}{ }{ }^{*}$ bo $v \rightarrow[\gamma]$ in $\hat{C}$, so $\alpha>\gamma \geqslant \beta$.
Now we exclude violations of (W2)(b) in the connect-through step, by showing that in $\hat{C}_{w_{2}}^{\left(w_{1}\right)}, \alpha>\beta$ for all newly created paths $u \underset{ \pm(u)}{ }[\alpha] \cdot \xrightarrow[ \pm(u)]{ }{ }_{\text {bo }}^{*} \cdot \rightarrow[\beta]$ with $u \neq w_{1}$ that stem from paths $u \underset{ \pm(u)}{\longrightarrow}[\alpha] \cdot \xrightarrow[\neq(u)]{\text { bo }_{0}} w_{1}$ and $w_{2} \xrightarrow[ \pm(u)]{*} \cdot \rightarrow[\beta]$ in $\hat{C}^{\prime}$. As $w_{2} G^{+} v$, there is a path $v \underset{\mathbf{t ( v )}}{ }[\gamma] \cdot \underset{\mathbf{t}(v)}{ }{ }^{*}$ bo $w_{2}$ in $\hat{C}^{\prime}$. We distinguish two cases.
CASE $1: u=v$. Then, by the level adaptation step, $\alpha=\gamma$. Since $u=v$, there is a path $v \underset{\mathbf{t ( v )}}{ }[\gamma] \cdot \xrightarrow[\mathbf{t}(v)]{\longrightarrow}{ }_{\text {bo }}^{*} w_{2} \xrightarrow[\neq(v)]{ }{ }_{\text {bo }}^{*} \cdot \rightarrow[\beta]$ in $\hat{C}^{\prime}$. By (W2)(b) for $\hat{C}^{\prime}, \gamma>\beta$.
CASE 2: $u \neq v$. Since $w_{1} G v$, there is a path $w_{1} \rightarrow_{\text {bo }}^{+} v$ in $\hat{C}$ and thus in $\hat{C}^{\prime}$. Suppose, toward a contradiction, that this path visits $u$. Then $u \underset{t(u)}{ }[\alpha] \cdot{\underset{t(u)}{ }}^{*}$ bo $w_{1} \rightarrow_{\text {bo }}^{+} u$, so $w_{1} G u$ in $\hat{C}^{\prime}$ and thus in $\hat{C}$. Then $w_{1}{ }_{d} G$ and $u \neq v$ imply $v \in u$, which together with $u \rightarrow_{\text {bo }}^{+} v$ yields a body step cycle between $u$ and $v$ in $\hat{C}$. This contradicts that (W1) holds in $\hat{C}$. Therefore $w_{1} \xrightarrow[\neq(u)]{ }{ }^{+} v$ in $\hat{C}^{\prime}$. We consider two cases.
 By (W2)(b) for $\hat{C}^{\prime}, \alpha>\beta$.

 $w_{1} \xrightarrow[t(u)]{\longrightarrow}+{ }_{\text {bo }}^{+} v \rightarrow[\gamma]$ in $\hat{C}^{\prime}$, by (W2)(b), $\alpha>\gamma>\delta_{k}>\cdots>\delta_{1}>\beta$. So $\alpha>\beta$.
To verify (W1) together with part (L2) of (W2)(a) for $\hat{\mathcal{C}}_{w_{2}}^{\left(w_{1}\right)}$, it suffices to show that $\hat{C}_{w_{2}}^{\left(w_{1}\right)}$ does not contain body step cycles. This can be verified analogously as for transformation I. That is, under the assumption of a body step cycle we can construct a path $w_{2} \rightarrow_{\text {bo }}^{+} w_{1}$ in $\hat{\mathcal{C}}$, which contradicts (C3) (as it contradicted (C1)).
To show part (L3) of (W2)(a) for $\hat{C}_{w_{2}}^{\left(w_{1}\right)}$, we can use part of the argumentation employed above for proving (W2)(b). It was demonstrated in particular that for every descends-in-loop-to path $u \underset{\mathbf{t}(u)}{\longrightarrow}[\alpha] \cdot \underset{\mathbf{t}(u)}{\longrightarrow}$ bo $x$ in $\hat{C}^{\prime \prime}$ there is a descends-in-loop-to path $\tilde{u} \underset{\mathbf{t}(\vec{u})}{ }\left[\gamma^{\prime}\right] \cdot \underset{\mathbf{t}(\vec{u})}{ }{ }^{*}$ bo $x$ with the same target $x$ in $\hat{C}$. This entails that if a descends-in-loop-to path in $\hat{C}^{\prime \prime}$ had $\sqrt{ }$ as target, then there were a descends-in-loop-to path in $\hat{C}$ with $\sqrt{ }$ as target, contradicting (L3) for the LLEE-witness $\hat{C}$. Hence $\hat{C}^{\prime \prime}$ must satisfy part (L3) of (W2)(a).
We conclude that the result of transformation III is again a LLEE-witness.


[^0]:    Report version.

[^1]:    ${ }^{1}$ Rutten [23] used this name for an analogous result on infinite streams [23]. The first author [12], and Kozen and Silva [19, 27] used it for the provable synthesis of regular expressions from their Brzozowski derivatives. The result here can be viewed as stating the provable synthesis of regular expressions from their partial derivatives (due to Antimirov [2]).

[^2]:    ${ }^{2}$ Confluence, and unique normalization, can be shown if a pruning operation is added that permits to drop transitions to deadlocking vertices.

[^3]:    ${ }^{3}$ We repeatedly pick vertices $v$ in the remaining LLEE-witness with entry step level $|v|_{\text {en }}$ (see in the text below) minimal.

