On Quasipolynomial Multicut-Mimicking Networks and Kernelization of Multiway Cut Problems

Magnus Wahlström
Department of Computer Science, Royal Holloway, University of London, UK
Magnus.Wahlstrom@rhul.ac.uk

Abstract

We show the existence of an exact mimicking network of $k^{O(\log k)}$ edges for minimum multicuts over a set of terminals in an undirected graph, where k is the total capacity of the terminals. Furthermore, if Small Set Expansion has an approximation algorithm with a ratio slightly better than $\Theta(\log n)$, then a mimicking network of quasipolynomial size can be computed in polynomial time. As a consequence of the latter, several problems would have quasipolynomial kernels, including Edge Multiway Cut, Group Feedback Edge Set for an arbitrary group, 0-Extension for integer-weighted metrics, and Edge Multicut parameterized by the solution and the number of cut requests. The result works via a combination of the matroid-based irrelevant edge approach used in the kernel for s-Multiway Cut with a recursive decomposition and sparsification of the graph along sparse cuts. The main technical contribution is a matroid-based marking procedure that we can show will mark all non-irrelevant edges, assuming that the graph is sufficiently densely connected. The only part of the result that is not currently constructive and polynomial-time computable is the detection of such sparse cuts.

This is the first progress on the kernelization of Multiway Cut problems since the kernel for s-Multiway Cut for constant value of s (Kratsch and Wahlström, FOCS 2012).

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1 Introduction

Graph separation questions are home to some of the most intriguing open questions in theoretical computer science. In approximation algorithms, the well-known *unique games conjecture* (UGC) has been central to the area for close to two decades, and is closely related to graph separation problems. Even more directly, the *small set expansion hypothesis*, proposed by Raghavendra and Steurer [31], roughly states that it is NP-hard to approximate the Small Set Expansion problem (SSE) up to a constant factor, where SSE is the problem of finding a small-sized set in a graph with minimum expansion. (More precise statements are given in Section 2.2.) Despite significant research, the best result available in polynomial time is an $O(\log n)$-approximation due to Räcke [30].

Another interesting notion from parameterized complexity is *kernelization*. Informally, a kernelization algorithm is a procedure that takes an input of a parameterized, usually NP-hard problem and *reduces* it to an equivalent instance of size bounded in the parameter, e.g., by discarding irrelevant parts of the input or transforming some part of the input into a smaller object with equivalent behaviour. For example, the seminal Nemhauser-Trotter theorem on the half-integrality of Vertex Cover [27] implies that an instance of Vertex Cover can be reduced to have at most $2k$ vertices, where k is the bound given on the solution size. On the flip side, Fortnow and Santhanam [10] and Bodlaender et al. [3] gave a framework to exclude the existence of a kernel of any polynomial size, under a standard complexity-theoretic conjecture. An extensive collection of upper and lower bounds for kernelization exists (see, e.g., the recent book of Fomin et al. [9]), but a handful of central "hard questions" remain unanswered. One of the most notorious is Multiway Cut.

Let $G = (V, E)$ be a graph and $T \subseteq V$ a set of terminals in G. An *(edge) multiway cut* for T in G is a set of edges $X \subseteq E$ such that no two terminals are connected in $G - X$, and Multiway Cut is the problem of finding a multiway cut of at most k edges, given a parameter k. The problem is FPT [24] and NP-hard for $|T| \geq 3$ [6]. Using methods from matroid theory, Kratsch and Wahlström [17] were able to show that if $|T| \leq s$, then Multiway Cut has a kernel with $O(k^{s+1})$ vertices, hence the problem has a polynomial kernel for every constant s. However, if $|T|$ is unbounded, the only known size bound for a kernel is $2^{O(k)}$, following from the FPT algorithm [24], and the question of whether Multiway Cut has a polynomial kernel in the general case is completely open.

We show a connection between kernelization of Multiway Cut-type problems and approximation algorithms for Small Set Expansion. Specifically, we show the existence of a kind of *mimicking network* for the problem, of size quasipolynomial in k; and if SSE has approximation algorithms slightly better than current state of the art, then it can be computed in polynomial time and Multiway Cut has a quasipolynomial kernel.

1.1 Mimicking networks and multiway cut sparsifiers

Although kernelization is most commonly described in terms of polynomial-time preprocessing as above, there is also a clear connection with *succinct information representation*. For example, consider a graph $G = (V, E)$ with a set of k terminals $T \subseteq V$. The pair (G, T) is referred to as a *terminal network*. A *mimicking network* for (G, T) is a graph $G' = (V', E')$ with $T \subseteq V'$ such that for any sets $A, B \subseteq T$, the min-cut between A and B in G and G' have the same value. A mimicking network of size bounded in k always exists, but the size of G' can be significant. The best known general upper bound is double-exponential in k [12, 15], and there is an exponential lower bound [20]. Better bounds are known for special graph classes, but even for planar graphs the best possible general bound has $2^{\Theta(k)}$ vertices [20, 14] (see also recent improvements by Krauthgamer and Rika [19]).

A related notion is *cut sparsifiers*, which solve the same task up to some approximation factor $q \geq 1$ [26, 21], typically $q = \omega(1)$ in the general case. We focus on mimicking networks; see Krauthgamer and Rika [19] for an overview of cut sparsifiers.

However, if we include the capacity of the set of terminals in the bound (and if edges have integer capacity), then significantly stronger results are possible. Chuzhoy [4] showed that if the total capacity of T is $\mathrm{cap}_G(T) = \sum_{t \in T} d(t) = k$, then there exists an $O(1)$-approximate cut sparsifier of size $O(k^3)$. Kratsch and Wahlström [17] sharpened this to an exact mimicking network with $O(k^3)$ edges, which furthermore can be computed in randomized polynomial time. This is particularly remarkable given that the network has to replicate the exact cut-value for exponentially many pairs (A, B). The network can be constructed via contractions on G.[1] This built on an earlier result that used linear representations of matroids to encode the sizes of all (A, B)-min cuts into an object using $\tilde{O}(k^3)$ bits of space [18], although this earlier version did not produce an explicit graph, i.e., not a mimicking network.

[1] The results of [17] are phrased in terms of vertex cuts, but the above follows easily from [17].

These results had significant consequences for kernelization. The succinct representation in [18] was used to produce a (randomized) polynomial kernel for the Odd Cycle Transversal problem, thereby solving a notorious open problem in parameterized complexity [18]; and the mimicking network of [17] brought further (randomized) polynomial kernels for a range of problems, in particular including Almost 2-SAT, i.e., the problem of satisfying all but at most k clauses of a given 2-CNF formula.

Similar methods are relevant for the question of separating a set of terminals into more than two parts. Let (G, T) be a terminal network, and let $\mathcal{T} = T_1 \cup \ldots \cup T_s$ be a partition of T. A *multiway cut for* $\mathcal{T}$ is a set of edges $X \subseteq E(G)$ such that $G - X$ contains no path between any pair of terminals $t \in T_i$ and $t' \in T_j$ for $t, t' \notin X$ and $i \neq j$. Let us define a *multicut-mimicking network* for (G, T) as a terminal network (G', T) where $T \subseteq V(G')$ and for every partition $\mathcal{T} = T_1 \cup \ldots \cup T_s$ of T, the size of a minimum multiway cut for $\mathcal{T}$ is identical in G and G'. (The term *multicut-mimicking*, as opposed to *multiway cut-mimicking*, is justified; see Section 2.1.) The minimum size of a multicut-mimicking network, in terms of $k = \mathrm{cap}_G(T)$, appears to lie at the core of the difficulty of the question of a polynomial kernelization of Multiway Cut. The kernel for s-Multiway Cut mentioned above builds on the computation of a mimicking network of size $O(k^{s+1})$ for partitions of T into at most s parts [17]. The kernel for s-Multiway Cut then essentially follows from considering the partition $\mathcal{T} = \{t_1\} \cup \ldots \cup \{t_s\}$ of a set T of $|T| = s$ terminals (along with known reduction rules bounding $\mathrm{cap}_G(T)$). We are not aware of any non-trivial lower bounds on the size of a multicut-mimicking network in terms of k; it seems completely consistent with known bounds that every terminal network (G, T) would have a multicut-mimicking network of size $\mathrm{poly}(k)$, even for partitions into an unbounded number of sets.

In this paper, we show that any terminal network (G, T) with $\mathrm{cap}_G(T) = k$ admits a multicut-mimicking network (G', T) where $|V(G')| = k^{O(\log k)}$; and furthermore, such a network could be computed in randomized polynomial time, given a polynomial number of queries to a sufficiently good approximation algorithm for a graph separation problem similar to Small Set Expansion. We also see a tradeoff between the quality of the approximation algorithm and the size of (G', T). In particular, if Small Set Expansion has an approximation algorithm with a ratio of $\alpha(n, k) = \log^{1-\varepsilon} n \cdot \log^{O(1)} k$ for some $\varepsilon > 0$, where k is the number of edges cut in the optimal solution, then (G', T) can be computed efficiently, with $|V(G')|$ being quasipolynomial in k. Such an algorithm goes beyond the bounds of what is currently known – namely, a ratio of $O(\log n)$ due to Räcke [30], improved for certain regimes by Bansal et al. [2] – but does not appear to be excluded by any established hardness conjecture. We also consider the existence result very interesting in its own right, and invite further study of capacity-based bounds for multicut-mimicking networks; in particular, whether a $\mathrm{poly}(k)$-sized multicut-mimicking network always exists. The results strongly suggest the existence of a quasipolynomial kernel for Edge Multiway Cut.

Flow sparsifiers. Finally, similarly to cut sparsifiers, there is a notion of a *flow sparsifier* of a terminal network (G, T). Here the goal is to approximately preserve the minimum congestion for any multicommodity flow on (G, T). Chuzhoy [4] showed flow sparsifiers with quality $O(1)$ and with $k^{O(\log \log k)}$ vertices, where k is the total terminal capacity; for further results on achievable bounds for flow sparsifiers, see [1, 7]. However, the notion is incomparable to multicut-mimicking networks, because even an exact flow sparsifier would be subject to the corresponding multicommodity flow-multicut approximation gap, which is $\Theta(\log k)$ in the worst case [11].

Further related work. The general approach of decomposing a graph along sparse cuts is well established; cf. Räcke [29] and follow-up work. For further applications of matroid tools to kernelization, see Hols and Kratsch [13], Kratsch [16], and Reidl and Wahlström [32].

1.2 Our results

More formally, we have the following.

▶ **Theorem 1.** *Let A be an approximation algorithm for* Small Set Expansion *with an approximation ratio of $\alpha(n,k) = O(\log^{1-\varepsilon} n \log^d k)$, where $\varepsilon > 0$, $d = O(1)$, and k is the number of edges cut in the optimal solution. Let (G,T) be a terminal network with $\mathrm{cap}_G(T) = k$. Then there is a set $Z \subseteq E(G)$ with $|Z| = k^{O(\alpha(n,k)\log k)}$ such that for every partition $\mathcal{T} = T_1 \cup \ldots \cup T_s$ of T, there is a minimum multiway cut X for $\mathcal{T}$ such that $X \subseteq Z$. Furthermore, Z can be computed in randomized polynomial time using calls to A.*

The precise requirement for the approximation algorithm is slightly relaxed from the above. We refer to the precise algorithm we need as a *sublogarithmic terminal expansion tester*; see Definition 5. Simplifying the statement a bit gives us the following.

▶ **Corollary 2.** *Let (G,T) be a terminal network with $\mathrm{cap}_G(T) = k$. The following holds.*

1. *There is a multicut-mimicking network for (G,T) with $k^{O(\log k)}$ edges.*
2. *If there is a sublogarithmic terminal expansion tester – in particular, if* Small Set Expansion *has an approximation ratio as in Theorem 1 – then a multicut-mimicking network of size quasipolynomial in k can be computed in randomized polynomial time.*

This would give us the following selection of conditional breakthrough results in kernelization. We refer to previous kernelization work [17, 32] for the necessary definitions.

▶ **Corollary 3.** *If there is a sublogarithmic terminal expansion tester, then the following problems have randomized quasipolynomial kernels.*

1. Edge Multiway Cut *parameterized by solution size.*
2. Edge Multicut *parameterized by the solution size and the number of cut requests.*
3. Group Feedback Edge Set *parameterized by solution size, for any group.*
4. Subset Feedback Edge Set *with undeletable edges, parameterized by solution size.*
5. 0-Extension *for integer-weighted graphs, parameterized by solution cost.*

Preliminaries. A *parameterized problem* is a decision problem where inputs are given as pairs $I = (X,k)$, where k is the *parameter*. A *polynomial kernelization* is a polynomial-time procedure that maps an instance (X,k) to an instance (X',k') where (X,k) is positive if and only if (X',k') is positive, and $|X'|, k' \leq g(k)$ for some function $g(k)$ referred to as the *size* of the kernel. A problem has a *polynomial kernel* if it has a kernel where $g(k) = k^{O(1)}$. We extend this to discuss *quasipolynomial kernels*, which is the case that $g(k) = k^{\log^{O(1)} k}$.

We use standard terminology from graph theory and parameterized complexity; see, e.g., [5, 9] for references.

2 Terminal separation notions

For a graph $G = (V,E)$ and sets $A, B \subseteq V$, we let $E_G(A,B) = \{uv \in E \mid u \in A, v \in B\}$. As shorthand for $S \subseteq V$ we also write $E(S) = E(S,S)$, $\partial_G(S) = E_G(S, V \setminus S)$, and $\delta_G(S) = |\partial_G(S)|$. The *total capacity* of a set of vertices S in a graph G is $\mathrm{cap}_G(S) := \sum_{v \in S} d(v)$. In all cases, we may omit the index G if understood from context.

2.1 Multicut-mimicking networks

Let $G = (V, E)$ be a graph and $T \subseteq V$ a set of terminals with $\mathrm{cap}_G(T) = k$. An *(edge) multiway cut* for T in G is a set of edges $X \subseteq E$ such that no two vertices in T are connected in $G - X$. More generally, let $\mathcal{T} = \{T_1, \ldots, T_r\}$ be a partition of T. Then an *(edge) multiway cut for $\mathcal{T}$ in G* is a set of edges $X \subseteq E$ such that in $G - X$ every connected component contains terminals from at most one part of $\mathcal{T}$. Hence a multiway cut for (G, T) is equivalent to a multiway cut for $(G, \{\{t\} \mid t \in T\})$. Furthermore, let $R \subseteq \binom{T}{2}$ be a set of pairs over T, referred to as *cut requests*. A *multicut for R in G* is a set of edges $X \subseteq E$ such that every connected component in $G - X$ contains at most one member of every pair $\{u, v\} \in R$. A *minimum multicut for R in G* is a multicut for R in G of minimum cardinality. Similarly, a *minimum multiway cut for $\mathcal{T}$ in G* is a multiway cut for $\mathcal{T}$ in G of minimum cardinality.

We define a *multicut-mimicking network* for T in G as a graph $G' = (V', E')$ such that $T \subseteq V'$ and such that for every set of cut requests $R \subseteq \binom{T}{2}$, the size of a minimum multicut for R is equal in G and in G'. We observe that this is equivalent to preserving the sizes of minimum multiway cuts over all partitions of T.

▶ **Proposition 4** (★[2]). *A graph G' with $T \subseteq V(G')$ is a multicut-mimicking network for T in G if and only if, for every partition $\mathcal{T}$ of T, the size of a minimum multiway cut for $\mathcal{T}$ is equal in G and in G'.*

As a slightly sharper notion, a *multicut-covering set* for (G, T) is a set $Z \subseteq E(G)$ such that for every set of cut requests $R \subseteq \binom{T}{2}$, there is a minimum multicut X for R in G such that $X \subseteq Z$. Note that a multicut-covering set Z is essentially equivalent to a multicut-mimicking network formed by contraction (contracting all edges of $E(G) \setminus Z$). Our main result in this paper is the existence of a multicut-covering set of size quasipolynomial in $k = \mathrm{cap}(T)$ in any undirected graph G. Furthermore, such a set can be computed in polynomial time, subject to the existence of certain approximation algorithms that we will make precise later in this section. The term is a generalization of a *cut-covering set*, used in previous work [17].

2.2 Graph separation algorithms

The central technical approximation assumption needed in this paper is the following. For a graph G with a set of terminals T, define the *T-capacity of S in G* as $\mathrm{cap}_T(S) = \mathrm{cap}_G(T \cap S) + \delta_G(S)$. Then we define the following notion.

▶ **Definition 5** (Sublogarithmic terminal expansion tester). *Let (G, T) be a terminal network with* $\mathrm{cap}_G(T) = k$. *A* terminal polynomial expansion tester (with approximation ratio α) *is a (possibly randomized) algorithm that, given as input (G, T) and an integer $c \in \mathbb{N}$, with $c = \Omega(\log k)$, does one of the following.*

1. *Either returns a set $S \subset V$ such that $N_G[S] \neq V(G)$ and $\mathrm{cap}_T(S) < |S|^{1/c}$,*
2. *or guarantees that for every set S with $\emptyset \subset (S \cap T) \subset T$ and $|S| \leq |V(G)|/2$ we have $\mathrm{cap}_T(S) \geq |S|^{1/c}/\alpha$.*

A sublogarithmic terminal expansion tester *is a terminal polynomial expansion tester with an approximation ratio $\alpha = O(\log^{1-\varepsilon} n \log^{O(1)} k)$ for some $\varepsilon > 0$. We say that (G, T) is (α, c)-*dense *if case 2 above applies, i.e., for every set S with $S \cap T \notin \{\emptyset, T\}$ and $|S| \leq |V(G)|/2$ we have $\mathrm{cap}_T(S) \geq |S|^{1/c}/\alpha$.*

[2] Proofs marked with ★ are found in the full version of the paper

The conditions can be relaxed somewhat. It is sufficient if the algorithm works with parameters $c = \Omega(\alpha \log k)$. It is also possible to put a lower bound on the size of sets S for which the guarantee needs to apply. However, these relaxed assumptions do not seem to make a difference for any algorithms we are aware of for the problem.

We note that such an algorithm would follow from a slightly improved approximation algorithm for Small Set Expansion. Let $G = (V, E)$ be a graph and $S \subseteq V$ a set of vertices. The *edge expansion* of S is $\Phi(S) := \frac{\delta(S)}{|S|}$. For a real number $\rho \in (0, 1/2]$, one also defines the *small set expansion* $\Phi_\rho(G) := \min_{S \subseteq V, |S| \leq \rho n} \Phi(S)$. In particular, for a value $s \in [n/2]$, $\Phi_{s/n}(G)$ denotes the worst (i.e., minimum) expansion among subsets of G of size at most s. A sufficiently good approximation algorithm for Small Set Expansion implies a sublogarithmic terminal expansion tester, as follows.

▶ **Lemma 6 (★).** *Assume that Small Set Expansion has a bicriteria approximation algorithm that on input (G, ρ) returns a set S with $|S| \leq \beta\rho n$ and $\Phi(S) \leq \alpha \cdot \Phi_\rho$, for some $\alpha, \beta \geq 1$. If $\alpha\beta = O(\log^{1-\varepsilon} n \log^{O(1)}(n \cdot \Phi_\rho))$, for some $\varepsilon > 0$, then there is a sublogarithmic terminal expansion tester with ratio $\Theta(\alpha\beta)$ (with $n \cdot \Phi_\rho$ replaced by k).*

Existing approximation algorithms do not meet this threshold; the best known results are an $O(\log n)$-approximation due to Räcke [30] and a bicriteria algorithm of Bansal et al. [2] which achieves a ratio of $O(\sqrt{\log n \log(1/\rho)})$. Unfortunately, the latter improvement is insufficient to make the analysis in the next section work. However, it seems clear that no existing hardness conjecture could possibly rule out the existence of such an algorithm. Furthermore, testing for (α, c)-denseness when $c = \Omega(\alpha \log k)$ corresponds to looking for significantly worse expanding sets than the regime usually focused on in the approximation literature. Hence we proceed with conditional results in the rest of the paper.

3 Multicut-covering sets

We now present the main result of the paper, namely the existence of quasipolynomial multicut-mimicking networks for terminal networks (G, T), and the conditional efficient computability of such objects given a sublogarithmic terminal expansion tester.

At a high level, the process works through recursive decomposition of the graph G across very sparse cuts, treating each piece $G[S]$ of the recursion as a new instance of multicut-covering set computation, where the edges of $\partial(S)$ are considered as additional terminals. The process repeatedly finds a single edge $e \in E(G)$ with a guarantee that for every set of cut requests $R \subseteq \binom{T}{2}$ there is a minimum multicut X for R in G such that $e \notin X$. We may then contract the edge e and repeat the process. Thus the end product is a multicut-mimicking network, and the edges that survive until the end of the process form a multicut-covering set.

In somewhat more detail, the process uses a novel variant of the representative sets approach, which was previously used in the kernel for s-Multiway Cut [17]. Refer to an edge e as *essential for* R, for some $R \subseteq \binom{T}{2}$, if every minimum multicut for R in G contains e, and *essential for* (G, T) if it is essential for R for some $R \subseteq \binom{T}{2}$. We use a representative sets approach to return a set of at most k^c edges which is guaranteed to contain every essential edge, *if* (G, T) is already (α, c)-dense, for an appropriate value $c = \Theta(\alpha \log k)$. On the other hand, if (G, T) is not (α, c)-dense, then (by careful choice of parameters) we can identify a cut through G which is sufficiently sparse that we can reduce the size of one side of this cut via a recursive call. This gives a tradeoff between the size of the resulting multicut-covering set and the denseness-guarantee we may assume through the approximation algorithm. The threshold for feasibility for this tradeoff is precisely the existence of a sublogarithmic terminal expansion tester.

3.1 Recursive replacement

We now present the recursive decomposition step in detail. Let (G, T) be a terminal network with $\mathrm{cap}_G(T) = k$. For a set $S \subseteq V$, we define the graph $G_S = G[N_G[S]] - E(S)$, i.e., G_S equals the graph $G[S]$ with the edges of $\partial(S)$ added back in. We also denote $T(S) = (T \cap S) \cup N_G(S)$ as the *terminals of* S. Under these definitions, the T*-capacity of* S *in* G has two equivalent definitions as $\mathrm{cap}_T(S) = \mathrm{cap}_{G_S}(T(S)) = \mathrm{cap}_G(T \cap S) + \delta_G(S)$. The *recursive instance at* S consists of the terminal network $(G_S, T(S))$. This is the basis of our recursive replacement procedure. Indeed, we show the following. Note that we consider $E(G_S) \subseteq E(G)$ in the following.

▶ **Lemma 7 (★).** *Let $(G_S, T(S))$ be the recursive instance at S for some $S \subseteq V(G)$. Let Z_S be a multicut-covering set for $(G_S, T(S))$ and let $e \in E(G_S) \setminus Z_S$. Then e is not essential for (G, T).*

Let us also briefly note the formal correctness of contracting a non-essential edge.

▶ **Proposition 8 (★).** *Let $e \in E(G)$ be a non-essential edge. Then for every $X \subseteq E(G)$ with $e \notin X$, and every partition $\mathcal{T}$ of T, X is a multiway cut for $\mathcal{T}$ in G if and only if it is a multiway cut for $\mathcal{T}$ in G/e. Furthermore, G/e is a multicut-mimicking network for (G, T), and any multicut-covering set $Z \subseteq E(G/e)$ for $(G/e, T)$ is also multicut-covering for (G, T).*

The process now works as follows. Recall that (G, T) is (α, c)-dense if $\mathrm{cap}_T(S) \geq |S|^{1/c}/\alpha$ for every set S with $S \cap T \neq \emptyset$ and $|S| \leq |V|/2$. The main technical result is a marking process that marks all essential edges for (G, T) on the condition that (G, T) is (α, c)-dense, and which marks at most k^c edges in total. In such a case, we are clearly allowed to select and contract any unmarked edge of G. Now, assume that (G, T) is not (α, c)-dense. Then by definition there exists a set $S \subset V$ such that $\mathrm{cap}_T(S) < |S|^{1/c}/\alpha$. If we can detect a set S such that $\mathrm{cap}_T(S) < |S|^{1/c}$, then we can recursively compute a multicut-covering set Z_S for $(G_S, T(S))$, consisting of at most $\mathrm{cap}_T(S)^c < |S|$ edges. By the above, we may again select any single edge $e \in E(G_S) \setminus Z_S$ and contract e in G. In either case, we replace G by a strictly smaller graph until $|E(G)| \leq k^c$, at which point we are done.

The two ingredients in the above are thus the marking process for (α, c)-dense graphs, which we present next, and the ability to distinguish the two cases, which is precisely the assumption of the existence of a sublogarithmic terminal expansion tester.

3.2 The (α, c)-dense case

Let us now focus on the marking procedure. Let a terminal network (G, T) with $\mathrm{cap}_G(T) = k$ and an integer c be given, and assume that $c = \Omega(\alpha \log k)$ for some α. We show a process that marks a set of at most k^c edges that contains every essential edge, assuming that (G, T) is (α, c)-dense. (A more precise bound on the relationship between c and α is given later, but the constant factors involved are not important to our main result.)

We will prove the following result. The proof takes up the rest of the subsection.

▶ **Lemma 9.** *Assume that (G, T) is (α, c)-dense where $c = \Omega(\alpha \log k)$. A multicut-covering set $Z \subseteq E(G)$ of size less than k^c can be computed in randomized polynomial time.*

The basis is the following. If (G, T) is (α, c)-dense then for every partition $\mathcal{T}$ of T, every minimum multiway cut X for $\mathcal{T}$, and every connected component H of $G - X$ except possibly the largest one, it holds that $\mathrm{cap}_T(V(H)) \geq |V(H)|^{1/c}/\alpha$. We also have

$$\sum_{H \in G - X} \mathrm{cap}_T(V(H)) = \mathrm{cap}_G(T) + 2|X| < 3k,$$

where the sum ranges over connected components H. This implies restrictions on the possible sizes of components of $G - X$, which will help in the marking process (as we shall see). Essentially, if too many components are too large, then the above sum will exceed $3k$ and we can conclude non-optimality of the corresponding multiway cut.

Finally, let us eliminate a silly edge case to assume $c \leq k$.

▶ **Lemma 10** (★). *If $c > k$ then a multicut-covering set of at most k^c edges can be marked deterministically.*

3.2.1 Matroid constructions

Before we show the marking procedure, we need some additional preliminaries. We refer to Oxley [28] and Marx [25] for further relevant background on matroids.

A *matroid* is a pair $M = (E, \mathcal{I})$ where $\mathcal{I} \subseteq 2^E$ is the *independent sets* of M, subject to the following axioms.

1. $\emptyset \in \mathcal{I}$;
2. if $B \in \mathcal{I}$ and $A \subseteq B$ then $A \in \mathcal{I}$; and
3. if $A, B \in \mathcal{I}$ with $|B| > |A|$ then there exists an element $x \in B \setminus A$ such that $A + x \in \mathcal{I}$.

A *basis* of M is a maximum independent set of M; the *rank* of M is the size of a basis.

Let A be a matrix, and let E label the columns of A. The *column matroid* of A is the matroid $M = (E, \mathcal{I})$ where $S \in \mathcal{I}$ for $S \subseteq E$ if and only if the columns indexed by S are linearly independent. A matrix A *represents* a matroid M if M is isomorphic to the column matroid of A. We refer to A as a *linear representation* of M.

We need three classes of matroids to build from. First, for a set E, the *uniform matroid* over E of rank r is the matroid $U(E, r) := (E, \{S \subseteq E \mid |S| \leq r\})$. Uniform matroids are representable over any sufficiently large field.

The second class is a *truncated graphic matroid*. Given a graph $G = (E, V)$, the *graphic matroid* of G is the matroid $M(G) = (E, \mathcal{I})$ where a set $F \subseteq E$ is independent if and only if F is the edge set of a forest in G. Graphic matroids can be deterministically represented over all fields. The *r-truncation* of a matroid $M = (E, \mathcal{I})$ for some $r \in \mathbb{N}$ is the matroid $M' = (E, \mathcal{I}')$ where $S \in \mathcal{I}'$ if and only if $S \in \mathcal{I}$ and $|S| \leq r$. Given a linear representation of M, over some field $\mathbb{F}$, a truncation of M can be computed in randomized polynomial time, possibly by moving to an extension field of $\mathbb{F}$ [25]. There are also methods for doing this deterministically [22], but the basic randomized form will suffice for us.

The final class is more involved. Let $D = (V, A)$ be a directed graph and $S \subseteq V$ a set of source vertices. A set $T \subseteq V$ is *linked to S in D* if there are $|T|$ pairwise vertex-disjoint paths starting in S and ending in T. Let $U \subseteq V$. Then $M(D, S, U) = (U, \{T \subseteq U \mid T \text{ is linked to } S \text{ in } D\})$ defines a matroid over U, referred to as a *gammoid*. Note that by Menger's theorem, a set T is dependent in M if and only if there is an (S, T)-vertex cut in D of cardinality less than $|T|$ (where the cut is allowed to overlap S and T). Like uniform matroids, gammoids are representable over any sufficiently large field, and a representation can be computed in randomized polynomial time [28, 25]. We will work over a variant of gammoids we refer to as *edge-cut gammoids*, which are defined as gammoids, except in terms of edge cuts instead of vertex cuts. Informally, for a graph $G = (V, E)$ and a set of source vertices $S \subseteq V$, the edge-cut gammoid of (G, S) is a matroid on a ground set of edges, where a set F of edges is independent if and only if it can be linked to S via pairwise edge-disjoint paths. However, we also need to introduce the "edge version" of *sink-only copies* of vertices, as used in previous work [17]. That is, we introduce a second set $E' = \{e' \mid e \in E\}$ containing copies of edges $e \in E$ which can only be used as the endpoints of linkages, not as initial or intermediate edges.

More formally, for a graph $G = (V, E)$ and a set of source vertices $S \subseteq E$ we perform the following transformation.

1. Subdivide every edge $e \in E$ by a new vertex z_e.
2. Let $p = \text{cap}_G(S)$. Inflate every vertex $v \in V$ into a twin class of $p + 1$ vertices (but do not inflate vertices z_e introduced in the previous step).
3. Replace every edge uv in the resulting graph by the two directed edges (u, v), (v, u), creating a directed graph D_G.
4. For every edge $e = uv \in E$, introduce a further new vertex z'_e, and create directed edges (u_i, z'_e) and (v_j, z'_e) for every copy u_i, v_j in D_G of the vertices u, v of G.

Slightly abusing notation, we let E refer to the vertices z_e in D_G, and we let E' refer to the vertices z'_e in D_G. The *edge-cut gammoid of* (G, S) is the gammoid $(D_G, \partial(S), E \cup E')$. Let us observe the resulting notion of independence.

▶ **Proposition 11.** *Let $G = (V, E)$ and $S \subseteq V$ be given. Let $M = (E \cup E', \mathcal{I})$ be the edge-cut gammoid of (G, S). Let $X \subseteq E \cup E'$ be given, and let $F = (X \cap E) \cup \{e \mid e' \in F \cap E'\}$. Then X is independent in M if and only if there exists a set $\mathcal{P}$ of $|X|$ paths linking X to S, where paths are pairwise edge-disjoint except that if $\{e, e'\} \subseteq X$ for some edge e, then two distinct paths in $\mathcal{P}$ end in e.*

We let $U(E, p)$ denote the uniform matroid of rank p on ground set $E(G)$, $M_G(p)$ the p-truncated graphic matroid of G, and $M(T)$ the edge-cut gammoid of (G, T).

If $M_1 = (E_1, \mathcal{I}_1)$ and $M_2 = (E_2, \mathcal{I}_2)$ are two matroids with $E_1 \cap E_2 = \emptyset$, then their *disjoint union* is the matroid $M_1 \uplus M_2 = (E_1 \cup E_2, \{I_1 \cup I_2 \mid I_1 \in \mathcal{I}_1, I_2 \in \mathcal{I}_2\})$. If M_1 and M_2 are represented by matrices A_1 and A_2 over the same field, then $M_1 \uplus M_2$ is represented by the matrix $A = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}$. We will define matroids M as the disjoint union over several copies of the base matroids $M(T)$, $M_G(p)$ and $U(E, p)$ defined above. In such a case, we refer to the individual base matroids making up M as the *layers* of M.

Representative sets

Our main technical tool is the representative sets lemma, due to Lovász [23] and Marx [25]. This result has been important in FPT algorithms [25, 8] and has been central to the previous kernelization algorithms for cut problems, including variants of Multiway Cut [17]. We also introduce some further notions.

▶ **Definition 12.** *Let $M = (E, \mathcal{I})$ be a matroid and $X, Y \in \mathcal{I}$. We say that Y* extends *X* in *M if $r(X \cup Y) = |X| + |Y|$, or equivalently, if $X \cap Y = \emptyset$ and $X \cup Y \in \mathcal{I}$. Furthermore, let $c = O(1)$ be a constant and let $\mathcal{Y} \subseteq \binom{E}{c}$. We say that a set $\hat{\mathcal{Y}} \subseteq \mathcal{Y}$* represents *$\mathcal{Y}$* in *$M$ if the following holds: For every $X \in \mathcal{I}$ for which there exists some $Y \in \mathcal{Y}$ such that Y extends X in M, then there exists some $Y' \in \hat{\mathcal{Y}}$ such that Y' extends X in M.*

▶ **Lemma 13** (representative sets lemma [23, 25])**.** *Let $M = (E, \mathcal{I})$ be a linear matroid represented by a matrix A of rank $r + s$, and let $\mathcal{Y} \subseteq \binom{E}{s}$ be a collection of independent sets of M, where $s = O(1)$. In time polynomial in the size of A and the size of $\mathcal{Y}$, we can compute a set $\hat{\mathcal{Y}} \subseteq \mathcal{Y}$ of size at most $\binom{r+s}{s}$ which represents $\mathcal{Y}$ in M.*

We will use the following *product form* of the representative sets lemma, with stronger specialized bounds. Assume that the rank of M is $r = r_1 + \ldots + r_c$, where r_i is the rank of layer i of M. Then Lemma 13 gives a bound on $|\hat{\mathcal{Y}}|$ as $\Theta((r_1 + \ldots + r_c)^c)$, but the following bound is significantly better when the layers of M have different rank.

▶ **Lemma 14** ([17]). *Let $M = (E, \mathcal{I})$ be a linear matroid, given as the disjoint union of c matroids $M_i = (E_i, \mathcal{I}_i)$, where M_i has rank r_i. Let $\mathcal{Y} \subseteq \binom{E}{c}$ be such that every set $Y \in \mathcal{Y}$ contains precisely one member in each layer M_i of M. Then the representative set $\hat{\mathcal{Y}} \subseteq \mathcal{Y}$ computed by the representative sets lemma will have $|\hat{\mathcal{Y}}| \leq \prod_{i=1}^{c} r_i$.*

3.2.2 The marking process

We are now ready to present the marking process.

Let $r = c - 2$. We define a process that marks edges of G in r passes, where each pass is a call to the representative sets lemma with a different matroid construction. Specifically, for each $i \in [r]$, define the following. The matroid M_i is the disjoint union of i copies of the edge-cut gammoid $M(T)$, one copy of $M_G(k^{r-i})$, and one copy of $U(E, k)$, where for $i = r$ we simply skip the copy of $M_G(k^0)$. We refer to the first i layers in M_i as the *gammoid layers* and the remaining as the *additional layers*. Note that a linear representation of M_i over some common field $\mathbb{F}$ can be computed in randomized polynomial time, since every layer of M_i can be represented over any sufficiently large field.

For each edge $e \in E$, let $t_i(e)$ be the set that contains a copy of z'_e in every gammoid layer, and a copy of e in every additional layer. Let $E_i = \{t_i(e) \mid e \in E\}$. For each pass $i \in [r]$, we compute a representative set $\hat{E}_i \subseteq E_i$ in the matroid M_i, and let $Z_i \subseteq E$ be the set of edges represented in $\hat{E}_i$. Let $Z = Z_1 \cup \ldots \cup Z_r$. We consider an edge $e \in E$ *marked* if $e \in Z$. We finish the description by observing the bound on the number of marked edges.

▶ **Lemma 15.** *The total number of marked edges is at most $rk^{r+1} < k^c$.*

Proof. By the product form of the representative sets lemma, $|Z_i| \leq k^{r+1}$ for every $i \in [r]$. ◀

Our main correctness condition for the marking is as follows. Consider a partition $\mathcal{T}$ of T and a corresponding minimum multiway cut $X \subseteq E$. Note that $|X| \leq k$ since $E(T, V)$ is a multiway cut for every partition. Say that X is *p-way plus q* if the p largest connected components of $G - X$ together cover all but q of the vertices. Say that X is *covered* if all edges essential for $\mathcal{T}$ are marked. We then have the following.

▶ **Lemma 16** (★). *If X is p-way plus k^{r-p} for $p \in [r]$, then X is covered in pass p above.*

Proof sketch. We define a set F_e such that $t_p(f)$ extends F_e if and only if $f = e$. By the existence of the set F_e, we then have a guarantee that $e \in Z_p$. To construct F_e, we follow [17] in the first p layers by letting F_e contain $X \cup \partial(T_i)$, where T_i is the set of vertices of T contained in the ith largest layer, thereby "blocking out" any edge contained in the p largest components from extending F_e. We use the two additional layers to block out edges containing in small components, respectively edges of $X - e$. Then by construction $t_p(f)$ fails to extend F_e for every edge $f \neq e$. Furthermore, as as in [17], we show that $t_p(e)$ extends F_e: if $t_p(e)$ fails to extend F_e in a gammoid layer, then this yields a "pushed solution" X_2 which is a minimum multiway cut for $\mathcal{T}$ with $e \notin X_2$, contradicting that e is essential for $\mathcal{T}$. In the additional layers, the argument for why e extends F_e is trivial. ◀

3.2.3 Correctness

We now argue that if (G, T) is (α, c)-dense for $c = \Omega(\alpha \log k)$ then every partition of T has a minimum multiway cut that is p-way plus k^{r-p} for some $p \in [r]$. For this, assume for a contradiction that for some partition $\mathcal{T}$ of T the minimum multiway cut X of $\mathcal{T}$ is not covered in any of the above passes. We will derive that $|X| > k$, contradicting that X is minimum. Assume that $G - X$ has p components, and let $n_1 \geq \ldots \geq n_p$ be the number of vertices in each component, sorted by size. The converse to Lemma 16 is the following.

▶ **Corollary 17.** *If X is not covered, then for every $i \in [r]$ it holds that $\sum_{j=i+1}^{p} n_j > k^{r-i}$.*

For $i \in [r]$, let us write $n_{\geq i} = \sum_{j=i}^{p} n_j$. Hence for each $i \in [r]$, $n_{\geq i+1} > k^{r-i}$.

Now, refer as previously to the vertex sets of the connected components of $G - X$ in order as $V_1, \ldots, V_p$, where $|V_i| = n_i$, $i \in [p]$. By the density assumption, for every $i \geq 2$, $\mathrm{cap}_T(V_i) \geq n_i^{1/c}/\alpha$. On the other hand, as previously noted, if X is minimum we have

$$\sum_{i=1}^{p} \mathrm{cap}_T(V_i) = \sum_{i=1}^{p} (\mathrm{cap}_G(T \cap V_i) + \delta(V_i)) = k + 2|X| \leq 3k. \tag{1}$$

It now remains to estimate the value of the following system:

$$\begin{array}{rrl} \min & \sum_{i=2}^{p} n_i^{1/c}/\alpha & \\ \text{s.t.} & \sum_{j=i+1}^{p} n_j > k^{r-i} & \forall i \in [r] \\ & \sum_{i=1}^{p} n_i = n & \\ & n_1 \geq \ldots \geq n_p \geq 0 & \end{array} \tag{2}$$

If we can determine that this value is greater than $3k$, then we will have derived a contradiction, showing that the cut X is covered. This is somewhat intricate, but not very difficult.

▶ **Lemma 18 (★).** *There is a $c = \Theta(\alpha \log k)$ such that the following holds: If (G, T) is (α, c)-dense, and if X is a multiway cut for some partition $\mathcal{T}$ of T such that X is not covered, then $|X| > k$.*

Proof sketch. Through concavity, one can show that the worst-case component sizes (i.e., the distribution n_i for which the system above achieves its minimum value) is when $n_i = k^{r-i+1}(1 - o(1))$ for every $i \geq 2$. The value of the system then becomes a geometric sum with ratio $k^{1/c} = 2^{\Theta(1/\alpha)}$, hence the total contribution is $\Theta((1/\alpha)k/(k^{1/c} - 1))$. Computing the asymptotics of the contributing factor $k^{1/c} - 1$ shows that it defeats $1/\alpha$, and establishes the result. ◀

3.3 Completing the result

By the above, every terminal network (G, T) that is (α, c)-dense for some $c = \Theta(\alpha \log k)$ has a multicut-covering set of at most k^c edges, which can be computed in randomized polynomial time. We extend the result to any (G, T), using a sublogarithmic terminal expansion tester.

▶ **Theorem 19** (Theorem 1 restated). *Let A be a sublogarithmic terminal expansion tester with ratio $\alpha(n, k)$. Let (G, T) be a terminal network with $\mathrm{cap}_G(T) = k$. There is a multicut-covering set $Z \subseteq E(G)$ with $|Z| \leq k^{O(\alpha(n,k) \log k)}$, which furthermore can be computed in randomized polynomial time using calls to A.*

Proof. Set $c = \Theta(\alpha \log k)$ as in Lemma 18. If $|E(G)| < k^c$ then return $Z = E(G)$, otherwise call A on (G, T, c). If (G, T) is (α, c)-dense, then Lemma 9 applies and we are done. Otherwise, let $S \subseteq V(G)$ be the set returned by A, and let $k_S = \mathrm{cap}_T(S)$. Let $(G_S, T(S))$ be the recursive instance at S, and note that $|V(G_S)| = |N_G[S]| < |V|$ and $|S| > k_S^c$ by definition of A. We may now proceed by induction on $|V|$ and assume that we can compute a multicut-covering set $Z_S \subseteq E(G_S)$ of size $|Z_S| < k_S^c$. To eliminate a corner case, if there is a vertex $v \in V(G_S)$ with $v \notin T(S)$ and $d_{G_S}(v) \leq 2$, then delete v if v is a leaf, otherwise contract one edge incident with v. Note that since $v \notin T(S)$ we have $d_G(v) = d_{G_S}(v)$ and $v \notin T$, hence these reduction rules are clearly correct. If this rule does not apply, there must be some edge $e \in E(G_S) \setminus Z_S$, and by construction e corresponds directly to an edge in G. Hence by

Prop. 8 we may contract e in G and repeat. This yields a graph G' with $|V(G')| < |V|$, hence by induction we can create a multicut-covering set Z for G', which is also a multicut-covering set of G by Prop. 8. Hence we can compute a multicut-covering set Z with $|Z| < k^c$. ◀

We observe the following consequences.

▶ **Corollary 20.** *Let (G, T) be a terminal network with $\mathrm{cap}_G(T) = k$. The following holds.*

1. *There is a multicut-mimicking network for (G, T) with $k^{O(\log k)}$ edges.*
2. *If there is a sublogarithmic terminal expansion tester – in particular, if* Small Set Expansion *has an approximation ratio as in Theorem 19 – then a multicut-mimicking network of size quasipolynomial in k can be computed in randomized polynomial time.*

Proof. The first is immediate using $\alpha(n, k) = 1$. For the second, all that remains is to clean up the value $|Z|$. For this, let $\alpha(n, k) \le \log^{1-\varepsilon} n \log^d k$ and $c = b\alpha \log k$, for some bounded values b, d, and first assume that $|Z| \ge |V(G)| = n$. Then

$$
\begin{aligned}
n \le |Z| < k^{b\alpha \log k} &\Rightarrow \\
\log n < b\alpha \log^2 k &\Rightarrow \\
\log n < b \log^{1-\varepsilon} n \log^{d+2} k &\Rightarrow \\
\log^{\varepsilon} n < b \log^{d+2} k &\Rightarrow \\
\log n < (b \log^{d+2} k)^{1/\varepsilon}&,
\end{aligned}
$$

hence $|Z| \le k^{\log^{O(1)} k}$, as promised. Otherwise, we contract all edges not present in Z and compute a new multicut-covering set Z' for the new system (G', T). Eventually, this process halts, and at this point we will have a multicut-covering set Z with $|Z| \le k^{\log^{O(1)} k}$ for some graph G'' created by contractions from G, and by Prop. 8 this set Z is also a multicut-covering set for (G, T). ◀

3.4 Kernelization extensions and consequences

As noted, we get the following consequences.

▶ **Corollary 21 (★).** *If there is a sublogarithmic terminal expansion tester, then the following problems have randomized quasipolynomial kernels.*

1. Edge Multiway Cut *parameterized by solution size.*
2. Edge Multicut *parameterized by the solution size and the number of cut requests.*
3. Group Feedback Edge Set *parameterized by solution size, for any group.*
4. Subset Feedback Edge Set *with undeletable edges, parameterized by solution size.*
5. 0-Extension *for integer-weighted graphs, parameterized by solution cost.*

Finally, as in [32], the latter result extends to "0-Extension sparsifiers" which hold independent of the choice of metric. Let us briefly recall some details. An instance of 0-Extension can be defined as a terminal network (G, T), a metric $\mu \colon D \times D \to \mathbb{R}^+$ for some label set D, and a partial labelling $\tau \colon T \to D$. The goal is to find $\lambda \colon V(G) \to D$ extending τ, to minimize the cost $\sum_{uv \in E(G)} \mu(\lambda(u), \lambda(v))$. We note that the "kernel" in the previous result can be constructed without needing access to μ or τ, i.e., it is valid for every metric μ and every partial labelling τ.

▶ **Theorem 22 (★).** *Let $G = (V, E)$ be an undirected, unweighted graph and $T \subseteq V$ a set of terminals, $|T| = r$. For any integer $p \in \mathbb{N}$, let $k = p + r$; there exists a set $Z \subseteq E$ with $|Z| = k^{O(\log k)}$ such that the following holds. For any metric $\mu \colon D \times D \to \mathbb{R}^+$ and any labelling $\tau \colon T \to D$, if there exists a labelling $\lambda \colon V \to D$ extending τ where $\lambda(u) \ne \lambda(v)$ for at most p edges $uv \in E$, then there exists such a labelling λ, of minimum cost among all such labellings, such that $\lambda(u) = \lambda(v)$ for every edge $uv \in E \setminus Z$.*

4 Discussion

We defined the notion of a *multicut-mimicking network*, and showed that every terminal network (G,T) with $k = \mathrm{cap}_G(T)$ admits one of size $k^{O(\log k)}$, which furthermore may be computable in randomized polynomial time, subject to the precise approximation guarantees available for a restricted variant of Small Set Expansion. The mimicking network can be constructed via contractions on G, i.e., it simply consists of a set of edges which form a *multicut-covering set*. As a consequence of such a result, a range of parameterized problems, starting from Edge Multiway Cut, would have quasipolynomial kernels. Unfortunately, the approximation guarantee required for this latter result appears to go just below the range of available guarantees from the literature.

A natural question is whether an appropriate approximation algorithm can be constructed. We note that an approximation ratio of $\mathrm{polylog}(k)$ for Small Set Expansion is sufficient, where $k = \delta(S)$ is the number of edges cut in the optimal solution S. We are not aware of approximation ratios in term of this parameter having been investigated. Also note that it is sufficient if the approximation algorithm has a running time quasipolynomial in k (but polynomial in n).

Another question is whether the existence of a polynomial-sized multicut-mimicking network can be established. Can such a result be excluded, even for the apparently more demanding situation of sparsifiers for 0-Extension instances (as in Theorem 22)?

We also have not investigated the vertex-deletion versions of these problems, which seem likely to bring significant additional difficulty (if such a generalization is possible).

In either case, the existence of a quasipolynomial multicut-covering set appears to rule out any possibility of a lower bound against the kernelizability of Edge Multiway Cut for any size better than quasipolynomial, given the nature of the lower bound results against kernelization. We hope, therefore (but dare not explicitly conjecture) that Edge Multiway Cut and related problems have quasipolynomial (randomized) kernels or better, unconditionally.

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