On The Relational Width of First-Order Expansions of Finitely Bounded Homogeneous Binary Cores with Bounded Strict Width*

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Abstract

The relational width of a finite structure, if bounded, is always (1, 1) or (2, 3). In this paper we study the relational width of first-order expansions of finitely bounded homogeneous binary cores where binary cores are structures with equality and some anti-reflexive binary relations such that for any two different elements a, b in the domain there is exactly one binary relation R with $(a, b) \in R$.

Our main result is that first-order expansions of liberal finitely bounded homogeneous binary cores with bounded strict width have relational width (2, MaxBound) where MaxBound is the size of the largest forbidden substructure, but is not less than 3, and liberal stands for structures that do not forbid certain finite structures of small size. This result is built on a new approach and concerns a broad class of structures including reducts of homogeneous digraphs for which the CSP complexity classification has not yet been obtained.

1 Introduction

An instance of the constraint satisfaction problem (CSP) consists of a number of variables and a number of local restrictions on variables called constraints. The question is whether there exists a global assignment to variables that satisfies all constraints. The CSP naturally generalizes SAT, expresses a number of other natural problems including k-coloring, solving equations over finite fields but, at least among theoreticians, is associated mainly to the question on dichotomy [17], i.e., is every right-hand side restriction $\text{CSP}(\mathbb{B})$ of the CSP in P or

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NP-complete? A relational structure \mathbb{B} , known also as a (constraint) language or a template, restricts the available constraints to these that can be modelled by relations in \mathbb{B} . It is known already for a while that the dichotomy for $CSP(\mathbb{B})$ over finite structures exists. Indeed, the Feder Vardi conjecture, on the existence of the dichotomy, has been confirmed by Zhuk [24] and independently by Bulatov [13].

The problem is that already a very simple scheduling problem with precedence constraints of the form (X < Y) cannot be properly expressed as $CSP(\mathbb{B})$ if the domain of $\mathbb B$ is finite and scheduling problems are what practitioners think of when they hear of the CSP. In order to express a richer class of problems including many scheduling problems as well as problems in spatial and temporal reasoning [7] one considers ω -categorical structures \mathbb{B} that, although infinite, share many nice properties with finite structures. In particular they admit a simple finite representation and the algebraic approach to the complexity of CSP which stands behind the both dichotomy proofs is also applicable in this context. (The precise definition of ω -categoricity as well as many other well-known notions used in the introduction are defined formally in the remainder of the paper, usually in Section 2.) It is no dichotomy to look for among all ω -categorical CSPs (CSPs over ω -categorical templates) [6]. Thus, one considers a subclass (first-order) reducts of (countably infinite) finitely bounded homogeneous structures for which a dichotomy is conjectured. In what follows we will call this conjecture the *infinite dichotomy conjecture*. In contrast to ω -categorical CSPs, all CSPs over reducts of finitely bounded homogeneous structures are in NP and more importantly, the infinite dichotomy conjecture is backed by an algebraic dichotomy [3] delineating algebras corresponding to structures with no non-trivial symmetries and such algebras with non-trivial symmetries. It is already known that algebras with no non-trivial symmetries correspond to NPcomplete problems. The *infinite tractability conjecture* states that a $CSP(\mathbb{B})$ is in P (tractable) always when the corresponding algebra contains some non-trivial symmetries (polymorphisms). A similar tractability conjecture concerning finite algebras was confirmed by Bulatov and Zhuk.

While in the finite case, the algorithm solving tractable $\text{CSP}(\mathbb{B})$ is based on two prevailing general algorithmic techniques: local-consistency methods [2] and the 'few subpowers' algorithm [19], the development of general algorithmic techniques and establishing the limits of their applicability in the infinite case are rather in their infancy. Two important exceptions are: an algebraic characterization of ω -categorical structures with bounded strict width [4, 5] and the lifting theorem [10] which lifts the tractability from finite CSP. Since it is already known that the tractability of some tractable reducts of (\mathbb{Q} ; <) a.k.a. temporal languages cannot be explained by the lifting theorem, the development of general algoritmic techniques for the CSPs within the scope of the infinite tractability conjecture and understanding the limits of their applicability seems inevitable. Natural research questions in this context concern local-consistency methods. Firstly, because the algebraic characterization of finite structures whose CSP may be solved in this way is considered to be an important step towards establishing the dichotomy. Secondly, because local consistency methods are ubiquitous in constraint solving, for instance, in the context of qualitative calculi in spatial and temporal reasoning [21]. In this paper we consider one of these natural questions.

A structure \mathbb{B} has bounded width if $CSP(\mathbb{B})$ is solvable by the local-consistency algorithm. Equivalently, $CSP(\mathbb{B})$ is solvable by an algorithm establishing (k, l)-minimiality for some natural numbers $k \leq l$. In this case we say that \mathbb{B} has relational width (k, l) and if such k, l exist that \mathbb{B} has bounded relational width [1, 12]. The relational width of \mathbb{B} proved to be a natural way to measure the amount of consistency needed to solve $CSP(\mathbb{B})$. In particular, it is known that if the relational width of a finite structure is bounded, then it is (1,1)or (2,3) [1]. This characterization is based on the algebraical characterization of finite structures with bounded (relational) width, and since its counterpart for reducts of finitely bounded homogeneous structures does not exist, it could be very hard to answer the question of what is the relational width of these structures. But as we already mentioned, there is such a characterization for structures with bounded strict width. A structure \mathbb{B} has strict width k if every partial solution to every (k, l)-minimal instance of $CSP(\mathbb{B})$ can be extended to a total solution. This notion is not only of theoretical interest [17] but also under the name local-to-global consistency has been studied in constraint solving in spatial and temporal reasoning, see e.g. [16].

In this paper we characterize the relational width of first-order expansions of liberal finitely bounded homogeneous binary cores with bounded strict width. All the definitions that are necessary to understand the result are given in the following subsection.

1.1 Results

We say that a structure \mathbb{A} over a relational signature (here assumed to be finite) is homogeneous if every local isomorphism between finite substructures of \mathbb{A} may be extended to an automorphism of \mathbb{A} . A structure \mathbb{A} over a signature τ is finitely bounded if there exists a finite set of finite τ -structures $\mathcal{F}_{\mathbb{A}}$ such that a finite structure Δ embeds into \mathbb{A} if and only if there is no Γ in $\mathcal{F}_{\mathbb{A}}$ that embeds into Δ .

All homogeneous graphs, classified in [20], and many homogeneous digraphs, classified in [15], are finitely bounded. In particular, it is known [18] that for any countable set of pairwise non-embedabble tournaments \mathcal{F} there exists a homogeneous digraph \mathbb{A} such that $\mathcal{F} = \mathcal{F}_{\mathbb{A}}$. Such homogeneous digraphs are known as Henson digraphs. In this paper we see homogeneous graphs and homogeneous digraphs over an extended signature and study these structures and many other as binary cores defined in what follows.

Binary Cores. We say that a structure \mathbb{A} over domain A is a *binary core* if its signature besides = contains only binary anti-reflexive relations R_1, \ldots, R_{κ} such that for any two different elements $a, b \in A$ there is exactly one R_i with $i \in [\kappa]$ such that $(a, b) \in R_i$.

Examples. A perfect example of a finitely bounded homogeneous binary core is a homogeneous graph seen over the signature $\{E, N, =\}$ where N contains all different pairs of elements which are not connected by an edge E, i.e., $(N(x, y) \equiv (\neg E(x, y) \land x \neq y))$. Other examples are homogeneous digraphs seen over the signature $\{\neg, N, =\}$ where \neg stands for an arc and N is a non-arc relation, i.e., $(N(x, y) \equiv \neg \curvearrowright (x, y) \land \neg \frown (y, x) \land x \neq y)$.

Liberal Structures. We restrict ourselves to these binary cores A that are additionally *liberal*, i.e., $\mathcal{F}_{\mathbb{A}}$ contains no finite structures of size 3, 4, 5, or 6. In particular any Henson digraph that forbids tournaments of size 7 or greater only, or a random graph \mathcal{G} seen over the extended signature is a liberal binary core. Indeed, all structures in $\mathcal{F}_{\mathcal{G}}$ are of size less than 3.

The Main Result. We write $\mathbb{L}_{\mathbb{A}}$ to denote the maximum of 3 and the size of the largest structure in $\mathcal{F}_{\mathbb{A}}$. A first-order expansion \mathbb{B} of \mathbb{A} is an expansion of \mathbb{A} such that all relations in \mathbb{B} have first-order definitions in \mathbb{A} . We are now in the position to formulate the main result of this paper.

Theorem 1 Let \mathbb{A} be a liberal finitely bounded homogeneous binary core and \mathbb{B} a first-order expansion of \mathbb{A} with bounded strict width. Then \mathbb{B} has relational width $(2, \mathbb{L}_{\mathbb{A}})$.

Examples of first-order expansions of the random graph with bounded strict width were given in [23].

Proposition 2 ([23]) Let the structure \mathbb{B} be a first-order expansion of the structure (A; E, N, =) where (A; E) is the random graph such that every relation in \mathbb{B} is pp-definable as a conjunction of clauses of the form:

$$(x_1 \neq y_1 \lor \cdots \lor x_k \neq y_1 \lor R(y_1, y_2) \lor y_2 \neq z_1 \lor \lor \cdots \lor y_2 \neq z_l),$$

where $R \in \{E, N\}$. Then A has bounded strict width.

Further, by Lemma 8 in [8] and Theorem 1 in [22], we have that an equality language, i.e., a first order expansion of $(\mathbb{N}; =, \neq)$ has bounded strict width if and only if all the extra relations are pp-definable by a conjunction of disjunctions of disequalities, i.e., clauses of the form:

$$(x_1 \neq y_1 \lor \cdots \lor x_k \neq y_k).$$

In order to prove Theorem 1 we show that all instances of $\text{CSP}(\mathbb{B})$ under consideration are simple in a particular sense. Roughly speaking, for a $(2, \mathbb{L}_{\mathbb{A}})$ minimal instance \mathcal{I} of $\text{CSP}(\mathbb{B})$ we construct a digraph $\mathcal{G}_{\mathcal{I}}$ over pairs ((v, x), C)where v, x are variables in the instance \mathcal{I} and $C \subsetneq A^2$ is a binary relation primitively-positively definable in \mathbb{B} . (Primitive positive (pp-)definitions are provided by primitive-positive (pp-)formulas which are first-order formulas built out of the conjunction, existential quantifiers and atomic formulae only.) There is an arc from $((v_1, x_1), C)$ to $((v_2, x_2), D)$ in $\mathcal{G}_{\mathcal{I}}$ if there is a constraint over a relation R whose scope (y_1, \ldots, y_k) contains v_1, x_1, v_2, x_2 and $R(y_1, \ldots, y_k)$ entails $(C(v_1, x_1) \Longrightarrow D(v_2, x_2))$ so that C and D are not the whole projections of $R(y_1, \ldots, y_k)$ to (v_1, x_1) and (v_2, x_2) , respectively. We say that \mathbb{B} is implicationally simple if every $\mathcal{G}_{\mathcal{I}}$ of every $(2, \mathbb{L}_{\mathbb{A}})$ -minimal instance \mathcal{I} of CSP(\mathbb{B}) is acyclic. We show that

- all implicationally simple first-order expansions of finitely bounded homogeneous binary cores A have relational width (2, L_A) (in Section 5) and
- all implicationally hard first-order expansions of liberal finitely bounded homogeneous binary cores have no bounded strict width (in Sections 3,4 and 6).

We would like to mention that not all first-order expansions of finitely bounded homogeneous binary cores with bounded strict width are implicationally simple. In [23] one can find the following.

Proposition 3 ([23]) The structure $\mathbb{B} = (A; E, N, =, R)$ where (A; E) is C_2^{ω} (two disjoint infinite cliques) and

$$R(x_1, x_2, x_3) \equiv ((E(x_1, x_2) \land N(x_2, x_3)) \lor (N(x_1, x_2) \land E(x_2, x_3)))$$

has bounded strict width.

Clearly, \mathbb{B} is not implicationally simple. Already $\mathcal{G}_{\mathcal{I}}$ for an instance \mathcal{I} with one constraint over relation R is not acyclic.

1.2 Related Work

In [23], the following was proved.

Theorem 4 ([23]) Let $\mathbb{A} = (A; E, N, =)$ be such that (A; E) is a homogeneous graph and \mathbb{B} a first-order expansion of \mathbb{A} with bounded strict width. Then \mathbb{B} has relational width $(2, \mathbb{L}_{\mathbb{A}})$.

Clearly, Theorem 1 generalizes Theorem 4 when \mathbb{A} is liberal but more importantly, the proof in [23] is based on the complexity classification of CSPs over first-order reducts of homogeneous graphs in [9], while the proof in this paper is based only on the assumptions that

- A is a liberal finitely bounded homogeneous binary core and that
- \mathbb{B} has bounded strict width.

In particular our result concerns all but finitely many finitely bounded Henson digraphs and many other structures for which CSP classifications has not been provided.

2 Preliminaries

We write [n] for $\{1, \ldots, n\}$ and when t is an n-tuple we write t[i] with $i \in [n]$ to denote the *i*-th value in t.

2.1 Structures under consideration

We consider here (countably infinite) finitely bounded homogeneous relational structures over domain A usually denoted by \mathbb{A} and their first-order expansions \mathbb{B} also over domain A. In particular we look at first-order expansions \mathbb{B} of (liberal) finitely bounded homogeneous binary cores \mathbb{A} . Recall that liberal means that $\mathcal{F}_{\mathbb{A}}$ contains no structures of size 3, 4, 5, and 6. For the sake of simplicity we write R to denote both a relational symbol in a signature of \mathbb{B} as well as the actual relation $R^{\mathbb{B}}$.

We will write $\operatorname{proj}_{i_1,\ldots,i_k} R$ for an *n*-ary relation R and $\{i_1,\ldots,i_k\} \subseteq [n]$ to denote the relation $R'(y_1,\ldots,y_k)$ defined by the formula

$$\exists x_1 \cdots \exists x_n \ R(x_1, \dots, x_n) \land \bigwedge_{j \in [k]} y_j = x_{i_j}.$$

We write $\operatorname{Aut}(\mathbb{A})$ to denote the set of automorphisms of \mathbb{A} . An orbit of a tuple t with values in A wrt. $\operatorname{Aut}(\mathbb{A})$ is the set $\{(\alpha(t[1]), \ldots, \alpha(t[n])) \mid \alpha \in \operatorname{Aut}(A)\}$. When \mathbb{A} is known from the context we simply say an orbit of a tuple instead of an orbit of a tuple wrt. $\operatorname{Aut}(\mathbb{A})$. We also say that O is an orbit if it is an orbit of some tuple. An orbital is an orbit of a tuple with two values.

All the structures under consideration are ω -categorical, i.e., their first-order theories have one countable model up to isomorphism. By the theorem proved independently by Ryll-Nardzewski, Engeler and Svenonius, a structure \mathbb{B} is ω categorical if and only if its automorphism group is oligomorphic, i.e., for every n the number of orbits of n-tuples is finite. Since \mathbb{A} is homogeneous, we have the following.

Observation 5 Let $\mathbb{A} = (A; R_1, \ldots, R_{\kappa}, =)$ be a binary core. Then R_i for all $i \in [\kappa]$ is an orbital.

Proof: The equality is clearly an orbital. Since \mathbb{A} is homogeneous we have that there is an automorphism $\alpha \in \operatorname{Aut}(\mathbb{A})$ such that $(\alpha(a_1), \alpha(a_2)) = (a_3, a_4)$ whenever $(a_1, a_2), (a_3, a_4) \in R_i$ and $i \in [\kappa]$. It follows that every R_i is a subset of some orbital wrt. Aut(\mathbb{A}). On the other hand, automorphism do not send $(a_1, a_2) \in R_i$ to $(a_3, a_4) \in R_j$ for $i \neq j$. The observation follows.

Further, since all structures under consideration have quantifier elimination, i.e., all first-order definable relations are definable without quantifiers, we have that every binary relation fo-definable in a binary core \mathbb{A} is a union of orbitals R_i with $i \in [\kappa]$. For a binary relation $C \subseteq A^2$ we will write C^{-1} to denote $(C^{-1}(x, y) \equiv C(y, x))$. We say that a binary relation C is anti-reflexive if it is contained in \neq or in other words if for all $(a, b) \in C$ we have that $a \neq b$. It happens that all liberal finitely bounded homogeneous binary cores pp-define \neq .

Observation 6 Let \mathbb{A} be a liberal finitely bounded homogeneous binary core. Then \mathbb{A} pp-defines \neq .

Proof: If \neq is an orbital wrt Aut(A), then we are done. Otherwise, there are at least two different orbitals $O_1, O_2 \subseteq \neq$. We claim that $(x_1 \neq x_2)$ is pp-defined by:

$$(\psi(x_1, x_2) \equiv (\exists x_0 \ O_1(x_0, x_1) \land O_2(x_0, x_2))).$$

Indeed, since O_1 and O_2 are different we have that an assignment $\mathbf{a} : \{x_1, x_2\} \to A$ such that $\mathbf{a}(x_1) = \mathbf{a}(x_2)$ does not satisfy ψ . On the other hand, since \mathbb{A} is liberal, for all anti-reflexive orbitals O_3 there exist element $a_0, a_1, a_2 \in A$ such that $(a_0, a_1) \in O_1, (a_0, a_2) \in O_2$ and $(a_1, a_2) \in O_3$. It implies that an assignment $\mathbf{a} : \{x_1, x_2\} \to A$ such that $\mathbf{a}(x_i) = a_i$ for $i \in [2]$ satisfies ψ . Since O_3 was chosen arbitrarily, we have that ψ is a definition of \neq .

In the paper, ternary and quaternary (4-ary) relations are of special interest. We will say that a tuple t = (t[1], t[2], t[3]) and t = (t[1], t[2], t[3], t[4]) are *OP*-tuples for some orbitals O, P if $(t[1], t[2]) \in O, (t[2], t[3]) \in P$ and $(t[1], t[2]) \in O, (t[3], t[4]) \in P$, respectively.

Further, we will say that a tuple is *constant* if all its values are the same and that is *non-constant* otherwise. A tuple is *injective* if all its values are pairwise different.

2.2 Entailment

A first-order formula $\varphi(x_1, \ldots, x_n)$ entails a first-order formula $\psi(x_1, \ldots, x_n)$ if the formula

$$(\forall x_1 \cdots \forall x_n \ (\varphi(x_1, \dots, x_n) \implies \psi(x_1, \dots, x_n)))$$

is valid. Further, we will say that an *n*-ary relation R entails a formula ψ over variables $\{x_1, \ldots, x_n\}$ if $R(x_1, \ldots, x_n)$ entails $\psi(x_1, \ldots, x_n)$.

We also say that an *n*-ary relation R entails no equalities if there are no different $i, j \in [n]$ such that R entails $(x_i = x_j)$.

2.3 CSP

A constraint **C** is a pair $((x_1, \ldots, x_k), R)$ where (x_1, \ldots, x_k) is the k-tuple of variables called also the *scope* of the constraint and R is a k-ary relation. We will write $\operatorname{proj}_{x_{i_1},\ldots,x_{i_l}} \mathbf{C}$ for the *projection* of a costraint $\mathbf{C} := ((x_1, \ldots, x_k), R)$ to a tuple of variables $(x_{i_1}, \ldots, x_{i_l})$ with $\{i_1, \ldots, i_l\} \subseteq [k]$. We will have that $\operatorname{proj}_{x_{i_1},\ldots,x_{i_l}} \mathbf{C}$ is the constraint $((x_{i_1},\ldots,x_{i_l}), R')$ where $R' = \operatorname{proj}_{i_1,\ldots,i_l} R$.

We study the problem $\text{CSP}(\mathbb{B})$ parametrized by first-order expansions of finitely bounded homogeneous structures. The instance \mathcal{I} of $\text{CSP}(\mathbb{B})$ is a set of constraints $((x_1, \ldots, x_k), R)$ such that R is a relation in \mathbb{B} . We say that \mathcal{I} is over variables \mathcal{V} if for every constraint $((x_1, \ldots, x_k), R)$ in \mathcal{I} we have that $x_1, \ldots, x_k \in \mathcal{V}$. The question in the problem $\mathrm{CSP}(\mathbb{B})$ is whether there exists a solution to \mathcal{I} , i.e., an assignment $\mathbf{s} : \mathcal{V} \to A$ to variables in \mathcal{I} such that for all constraints $((x_1, \ldots, x_k), R)$ we have $(\mathbf{s}(x_1), \ldots, \mathbf{s}(x_k)) \in R$.

Let $W \subseteq \mathcal{V}$. An assignment $\mathbf{a} : W \to A$ is a partial solution to \mathcal{I} if \mathbf{a} satisfies all projections of constraints in \mathcal{I} to variables in W. We will sat that \mathcal{I} entails no equalities if no relations in the constraints of \mathcal{I} do.

2.4 The universal-algebraic approach

We say that an operation $f: A^n \to A$ is a polymorphism of an *m*-ary relation Riff for all *m*-tuples $t_1, \ldots, t_n \in R$, it holds that the tuple $(f(t_1[1], \ldots, t_n[1]), \ldots, f(t_1[m], \ldots, t_n[m]))$ is also in R. We will write $f(t_1, \ldots, t_n)$ as a shorthand for the expression $(f(t_1[1], \ldots, t_n[1]), \ldots, f(t_1[m], \ldots, t_n[m]))$. An operation fis a polymorphism of \mathbb{A} if it is a polymorphism of every relation in \mathbb{A} . If f: $A^n \to A$ is a polymorphism of \mathbb{A} , R, we say that f preserves \mathbb{A}, R . A set of polymorphisms of an ω -categorical structure \mathbb{A} forms an algebraic object called an oligomorphic locally closed clone [4], which in particular contains an oligomorphic permutation group [14].

Recall that a first-order formula is a *primitive-positive formula (pp-formula)* if it is built out of conjunction, existential quantifiers and atomic formulae only. There is a deep connection between the polymorphisms of a structure and the relations pp-definable in that structure.

Theorem 7 ([11]) Let \mathbb{A} be a countable ω -categorical structure. Then R is preserved by the polymorphisms of \mathbb{A} if and only if it has a primitive-positive definition in \mathbb{A} .

We say that a set of operations F generates a set of operations G if every $g \in G$ is in the smallest locally-closed clone containing F. An operation f of an oligomorphic clone F is called *oligopotent* if $\{g\}$ where $g(x) := f(x, \ldots, x)$ is generated by the permutations in F. We say that a k-ary operation f over domain A is a quasi near-unanimity operation (qnu-operation) if

$$f(y, x, \dots, x) = f(x, y, x, \dots, x) = \dots =$$
$$\dots = f(x, \dots, x, y) = f(x, \dots, x)$$

for all $x, y \in A$.

2.5 Widths and Minimality

We will now give a formal definition of a (k, l)-minimal instance.

Definition 8 We say that an instance \mathcal{I} over \mathcal{V} of $CSP(\mathbb{B})$ is (k, l)-minimal with $k \leq l$ if both of the following hold:

 every subset of at most l variables in V is contained in a scope of some constraint in I and • for every at most k-element subset of variables $X = \{x_1, \ldots, x_k\} \subseteq \mathcal{V}$ and any two constraints $C_1, C_2 \in \mathcal{I}$ whose scopes contain X the projections $proj_{x_1,\ldots,x_k} C_1$ and $proj_{x_1,\ldots,x_k} C_2$ are the same.

We say that an instance \mathcal{I} of the CSP is *non-trivial* if it does not contain a constraint $((x_1, \ldots, x_k), \emptyset)$. Otherwise, \mathcal{I} is *trivial*.

Set $k \leq l$. Clearly not every instance \mathcal{I} over variables \mathcal{V} of $\operatorname{CSP}(\mathbb{B})$ for \mathbb{B} over domain A is (k, l)-minimal, however, the algorithm that obtains an equivalent (k, l)-minimal instance is straightforward and works in time $O(|\mathcal{V}|^m)$ where m is the maximum of l and the largest arity in the signature of \mathbb{B} . Indeed, it is enough to introduce a new constraint $((x_1, \ldots, x_l), A^l)$ for all pairwise different variables $x_1, \ldots, x_l \in \mathcal{V}$ to satisfies the first condition. Then the algorithm removes tuples (orbits) from the relations in constraints in the instance as long as the second condition is not satisfied. It is widely known and easy to prove that an instance \mathcal{I}' of the CSP obtained by the described algorithm is equivalent, i.e., has the same set of solution, to the orginal instance \mathcal{I} . In particular we have that if \mathcal{I}' is trivial, then \mathcal{I} has no solutions. Under a natural assumption that \mathbb{B} contains all at most l-ary relations pp-definable in \mathbb{B} , we have that \mathcal{I}' is an instance of CSP(\mathbb{B}). From now on this assumption will be in effect.

Definition 9 A relational structure \mathbb{B} has relational width (k, l) if every (k, l)minimal instance \mathcal{I} of \mathbb{B} has a solution iff it is non-trivial.

A relational structure \mathbb{B} has bounded relational width if it has relational width (k,l) for some natural numbers $k \leq l$.

In this paper we mainly look at $(2, \mathbb{L}_{\mathbb{A}})$ -minimal instances \mathcal{I} of $CSP(\mathbb{B})$ for first-order expansions \mathbb{B} of finitely bounded homogeneous binary cores \mathbb{A} . For such instances \mathcal{I} and $x, y \in \mathcal{V}$ we will write $\mathcal{I}_{x,y}$ to denote the projection of any constraint in \mathcal{I} to the variables (x, y).

We now turn to strict width.

Definition 10 We say that \mathbb{B} has strict width k if there exists l such that every partial solution of every (k, l)-minimal instance of $CSP(\mathbb{B})$ may be extended to a total solution.

The following theorem provides a characterization of bounded strict width that we use intensively in this paper.

Theorem 11 [5, 4] Let \mathbb{B} be an ω -categorical language. Then the following are equivalent.

- 1. \mathbb{B} has strict width k.
- B has an oligopotent (k+1)-ary quasi near-unanimity operation as a polymorphism.

Observe that in order to show that some structure \mathbb{B} has no bounded strict width it is enough to show that they are not preserved by any oligopotent quu-operations or, by Theorem 7 to pp-define a structure \mathbb{B}' of which we already know that has no bounded strict width.

3 Critical Ternary Relations

We define a family of structures which is the main source of 'infinite' strict width, i.e., whenever we want to show that some structure does not have bounded strict width we pp-define a *critical ternary relation*.

Definition 12 We say that a relation R is a critical ternary relation over $(\mathbb{B}, C_1, C_2, D_1, D_2)$ if all of the following hold:

- R, C_1, C_2, D_1, D_2 are pp-definable in \mathbb{B} ,
- C_1 and C_2 are disjoint and contained in $proj_{1,2}R$
- D_1 and D_2 are disjoint and contained in $proj_{2,3} R$,
- both C_1, D_1 are anti-reflexive,
- either both C_2, D_2 are anti-reflexive or both are =,
- R entails $(C_1(x_1, x_2) \implies D_1(x_2, x_3))$
- R entails $(D_1(x_2, x_3) \implies C_1(x_1, x_2))$
- R contains both

$$(R_1(x_1, x_2, x_3) \equiv (C_1(x_1, x_2) \land D_1(x_2, x_3)))$$

and

$$(R_2(x_1, x_2, x_3) \equiv (C_2(x_1, x_2) \land D_2(x_2, x_3)))$$

Before we turn to the main result of this subsection we need two observations.

Observation 13 Let \mathbb{A} be a liberal finitely bounded homogeneous binary core and R a critical ternary relation over $(\mathbb{B}, C_1, C_2, D_1, D_2)$ for some first-order expansion \mathbb{B} of \mathbb{A} . Let $k \in \mathbb{N}$ and $I, J \subseteq [k] \setminus \{m\}$ for some $m \in [k]$ be disjoint subsets of indices such that $I \cup J \cup \{m\} = [k]$ and $w, u \in A^k$ such that all of the following hold:

- $(w[i], u[i]) \in C_1$ for all $i \in I$,
- $(w[m], u[m]) \in C_2$, and
- $(w[i], u[i]) \in C_2$ for all $i \in J$.

Then there exists $v \in A^k$ such that all of the following hold:

- $(w[i], u[i], v[i]) \in R_1 \text{ for all } i \in I,$
- $(w[m], u[m], v[m]) \in \{(x_1, x_2, x_3) \in A^3 \mid C_2(x_1, x_2) \land D_1(x_2, x_3) \land x_1 \neq x_3\}, and$
- $(w[i], u[i], v[i]) \in R_2$ for all $i \in J$.

Proof: Starting with a substructure of the structure A induced by the elements $w[1], \ldots, w[k], u[1], \ldots, u[k]$ satisfying the conditions in the formulation of the lemma we will move around the structure A to find $v[1], \ldots, v[k]$ so that the structure induced by all elements in w, u and v satisfy all the requirements. Assume we are done for $v[1], \ldots, v[i]$ for some $i \in [k]$ and consider v[i + 1]. Then one of three requirements has to be satisfied by v[i + 1]. The first case to consider is where $(i + 1) \in I$ and we require $(w[i + 1], u[i + 1], v[i + 1]) \in R_1$. Since A is liberal, there are three pairwise different elements a_1, a_2, a_3 in A such that (a_1, a_2) are in the same orbital as (w[i + 1], u[i + 1]) and (a_2, a_3) is in some (actually in any) orbital contained in D_1 . Since A is homogeneous there is an automorphism α sending (a_1, a_2) to (w[i + 1], u[i + 1]). Then we take v[i + 1] to be $\alpha(a_3)$. When i + 1 = m then either D_2 is anti-reflexive and we proceed as in one of the cases above. Indeed, either $C_2, D_2 \subseteq \neq$ or both C_2, D_2 equal =.

We also need to consider a situation from the previous observation where u, v are given and one looks for w.

Observation 14 Let \mathbb{A} be a liberal finitely bounded homogeneous binary core and R a critical ternary relation over $(\mathbb{B}, C_1, C_2, D_1, D_2)$ for some first-order expansion \mathbb{B} of \mathbb{A} . Let $k \in \mathbb{N}$ and $I, J \subseteq [k] \setminus \{m\}$ for some $m \in [k]$ be disjoint subsets of indices such that $I \cup J \cup \{m\} = [k]$ and $u, v \in A^k$ such that

- $(u[i], v[i]) \in D_1$ for all $j \in I$,
- $(u[m], v[m]) \in D_2$, and
- $(u[i], v[i]) \in D_2$ for all $i \in J$.

Then there exists $w \in A^k$ such that all of the following hold:

- $(w[j], u[j], v[j]) \in R_1$ for all $j \in I$,
- $(w[m], u[m], v[m]) \in \{(x_1, x_2, x_3) \in A^3 \mid C_1(x_1, x_2) \land D_2(x_2, x_3) \land x_1 \neq x_3\},\$ and
- $(w[j], u[j], v[j]) \in R_2$ for all $j \in J$.

We are now in the position to prove that critical ternary relations ppdefinable in first-order expansions of liberal finitely bounded homogeneous cores have no bounded strict width.

Proposition 15 Let \mathbb{A} be a liberal finitely bounded homogeneous binary core and R a critical ternary relation over $(\mathbb{B}, C_1, C_2, D_1, D_2)$ for some first-order expansion \mathbb{B} of \mathbb{A} . Then both \mathbb{B} and R do not have bounded strict width. **Proof:** By Theorem 11, it is enough to prove that R is not preserved by any oligopotent qnu-operation. Suppose \mathbb{B} is preserved by a k-ary oligopotent quasi near-unanimity operation f. The essence of the proof of the proposition is in the following observation.

Let C be a binary relation. We will say that two k-tuples t_1, t_2 are C-connected on a coordinate $i \in [k]$ if $(t_1[i], t_2[i]) \in C$.

Observation 16 For all $i \in \{0, ..., k\}$ we have that both of the following hold.

- 1. For all $(w, u) \in (A^k)^2$ such that u is constant, (w, u) are C_1 -connected on (k - i) coordinates and C_2 -connected on i coordinates, we have that $(f(w), f(u)) \in C_1$.
- 2. For all $(u,v) \in (A^k)^2$ such that u is constant, (u,v) are D_1 -connected on (k-i) coordinates and D_2 -connected on i coordinates, we have that $(f(u), f(v)) \in D_1$.

Proof: The proof goes by the induction on *i*. In the base case where i = 0, the claim follows by the fact that C_1, D_1 are pp-definable in \mathbb{B} .

For the induction step, suppose first that Item 2 fails for some i > 1 and J with |J| = i, i.e., there exist $(u, v) \in (A^k)^2$ such that u is constant, (u, v) are D_1 -connected on $(k \setminus [J])$ and D_2 -connected on J and $(f(u), f(v)) \notin D_1$. Let $m \in J$. We set $J' = J \setminus \{m\}$ and $I' = [k] \setminus J$. It follows that [k] is a disjoint union of $I', \{m\}$ and J' and we have all of the following:

- $(u[j], v[j]) \in D_1$ for all $j \in I'$,
- $(u[m], v[m]) \in D_2$, and
- $(u[j], v[j]) \in D_2$ for all $j \in J'$

By Observation 14 there exists $w \in A^k$ such that all of the following hold:

- $(w[j], u[j], v[j]) \in R_1$ for all $j \in I'$,
- $(w[m], u[m], v[m]) \in \{(x_1, x_2, x_3) \in A^3 \mid C_1(x_1, x_2) \land D_2(x_2, x_3) \land x_1 \neq x_3\}$, and
- $(w[j], u[j], v[j]) \in R_2$ for all $j \in J'$.

By the induction hypothesis, it holds that $(f(w), f(u)) \in C_1$. Now, since \mathbb{A} is liberal and $C_1, D_1 \subseteq \neq$ there exist pairwise different b_1, b_2, b_3 such that (b_1, b_3) are in the same orbital as $(w[m], v[m]), (b_1, b_2) \in C_1$ and $(b_2, b_3) \in D_1$. Since \mathbb{A} is homogeneous, there is an automorphism $\alpha \in \mathbb{A}$ sending (b_1, b_3) to (w[m], v[m]). Let a be such that $a = \alpha(b_2)$ and $u' \in A^k$ such that u'[m] = a and u'[j] = u[j] whenever $j \neq m$. Observe that $(w[m], a, v[m]) \in R_1$.

Since for all $j \in [k]$ we have that $(w[j], u'[j], v[j]) \in R_1$ or $(w[j], u'[j], v[j]) \in R_2$, it follows that $(w[j], u'[j], v[j]) \in R$ for all $j \in [k]$. The relation R is preserved by f. Thus, $(f(w), f(u'), f(v)) \in R$. The tuple u is constant,

f is an oligopotent quu-operation, and hence f(u') = f(u). It implies that $(f(w), f(u')) \in C_1$. The relation R entails $(C_1(x_1, x_2) \implies D_1(x_2, x_3))$, and hence $(f(u'), f(v)) \in D_1$. Since f(u') = f(u), it follows that $(f(u), f(v)) \in D_1$. It contradicts the assumption that $(f(u), f(v)) \notin D_1$ and proves that Item 2 holds for the induction step.

The proof that Item 1 goes through the induction step is analogous to the proof for Item 2 with a difference that we use Observation 13 instead of Observation 14. It completes the proof of the induction step and the observation. \Box

Observation 16 implies $f(D_2, \ldots, D_2) = D_1$. It contradicts the fact that D_2 is pp-definable in \mathbb{B} and completes the proof of the proposition.

4 Efficient Entailment and Implications

In this section, we first provide the definition of an implication, which is needed to formally define implicationally simple and implicationally hard structures. Then, in the following subsections, we prove certain preliminary results which are then used in the proof of the main theorem. In particular, in Section 4.1, we show some auxiliary results that will be used to pp-define critical ternary relations out of a pair of complementary implications. In Section 4.2, we show that some implications are not pp-definable in first-order expansions of liberal finitely bounded homogeneous binary cores with bounded strict width, in Section 4.3 how to compose implications and in Section 4.4 that out of a pair of complementary implications we can always pp-define an implication of a very concrete form called a complete implication.

We start with a definition of *efficient entailment* which is a non-standard version of entailment from Section 2 and concerns ternary and quaternary relations. First, we look into the ternary ones.

Definition 17 Let R be a ternary relation and C_1, D_1 binary relations. We say that R efficiently entails:

$$(C_1(x_i, x_j) \implies D_1(x_k, x_l))$$

with $i, j, k, l \in [3]$ if both of the following hold:

- $R(x_1, x_2, x_3)$ entails $(C_1(x_i, x_j) \implies D_1(x_k, x_l))$,
- $C_1 \subsetneq proj_{i,j} R$ and $D_1 \subsetneq proj_{k,l} R$.

We have a similar definition for quaternary relations.

Definition 18 Let R be a quaternary relation and C_1, D_1 binary relations. We say that R efficiently entails:

$$(C_1(x_i, x_j) \implies D_1(x_k, x_l))$$

with $i, j, k, l \in [4]$ if both of the following hold:

- $R(x_1, x_2, x_3, x_4)$ entails $(C_1(x_i, x_j) \implies D_1(x_k, x_l)),$
- $C_1 \subsetneq proj_{i,j} R \text{ and } D_1 \subsetneq proj_{k,l} R.$

Next we define a ternary 'implication'.

Definition 19 Let $L, P \in \{\leftarrow, \rightarrow\}$. We say that a ternary relation R is a ternary $(\mathbb{B}, C, D, C_1, D_1, L, P)$ -implication if all of the following hold:

- 1. R entails no equalities,
- 2. all R, C_1, D_1 are pp-definable in \mathbb{B} ,
- 3. $proj_{i,j} R = C$ where (i,j) = (1,2) if $L \Longrightarrow and (i,j) = (2,1)$ if $L \Longrightarrow (i,j) = (2,1)$
- 4. $proj_{k,l} R = D$ where (k,l) = (2,3) if $L \rightarrow and (k,l) = (3,2)$ if $L \rightarrow (4,2)$
- 5. R efficiently entails $(C_1(x_i, x_j) \implies D_1(x_k, x_l))$.

We now provide a similar definition for quaternary relations.

Definition 20 Let $L, P \in \{\leftarrow, \rightarrow\}$. A quaternary relation R is a quaternary $(\mathbb{B}, C, D, C_1, D_1, L, P)$ -implication if all of the following hold:

- 1. R entails no equalities,
- 2. all R, C_1, D_1 are pp-definable in \mathbb{B} ,
- 3. $proj_{i,j} R = C$ where (i,j) = (1,2) if $L \Longrightarrow and (i,j) = (2,1)$ if $L \Longrightarrow (i,j) = (2,1)$
- 4. $proj_{k,l} R = C$ where (k,l) = (3,4) if $L \Longrightarrow and (k,l) = (4,3)$ if $L \Longrightarrow (4,3)$
- 5. R efficiently entails $(C_1(x_i, x_j) \implies D_1(x_k, x_l))$.

For the sake of succinctness we say that R is a $(\mathbb{B}, C, D, C_1, D_1, L, P)$ implication without specifying that it is ternary or quaternary if not necessary, or that a relation R is a (ternary or quaternary) $(\mathbb{B}, C, D, C_1, D_1)$ -implication if it is a $(\mathbb{B}, C, D, C_1, D_1, L, P)$ -implication for some $L, P \in \{\leftarrow, \rightarrow\}$ or that Ris a (ternary or quaternary) (L, P)-implication if it is a $(\mathbb{B}, C, D, C_1, D_1, L, P)$ implication for some $\mathbb{B}, C, D, C_1, D_1$.

Example. Let the relation R be a critical ternary relation over $(\mathbb{B}, C_1, C_2, D_1, D_2)$. Observe that the relation R is a $(\mathbb{B}, \operatorname{proj}_{1,2} R, \operatorname{proj}_{2,3} R, C_1, D_1, \rightarrow, \rightarrow)$ -implication and that

$$R'(x_1, x_2, x_3) \equiv R(x_3, x_2, x_1)$$

is a $(\mathbb{B}, \operatorname{proj}_{3,2} R, \operatorname{proj}_{2,1} R, D_1^{-1}, C_1^{-1}, \rightarrow, \rightarrow)$ -implication.

4.1 From Implications to Critical Ternary Relations

In order to define critical ternary relations, for instance, out of a pair (or a bunch) of implications we provide pp-definitions in many steps using often similar constructions. For the sake of succinctness, we will use certain shorthands.

Definition 21 We write $R_3 := R_1 \bowtie R_2$ for a ternary relation $R_3(x_1, x_2, x_3)$ defined out of two ternary relations R_1 and R_2 as follows

$$\exists y \ R_1(x_1, x_2, y) \land R_2(y, x_2, x_3) \tag{1}$$

or a quaternary relation $R_3(x_1, x_2, x_3, x_4)$ defined out of two quaternary relations R_1, R_2 as follows:

$$\exists y_1 \exists y_2 \ R_1(x_1, x_2, y_1, y_2) \land R_2(y_2, y_1, x_3, x_4) \tag{2}$$

We also write $R_1 \bowtie_3 R_2$ for a ternary relation $R_3(x_1, x_2, x_3)$ defined out of two quaternary relations R_1, R_2 as follows:

$$\exists y_1 \exists y_2 \ R(x_1, x_2, y_1, y_2) \land R(y_2, y_1, x_2, x_3) \tag{3}$$

The following observation provides us with some insight into the structure of $R_1 \bowtie R_2$ on the condition that we provide some assumptions on R_1, R_2 .

Observation 22 Let R_1, R_2 be both ternary or both quaternary relations fodefinable in a liberal finitely bounded homogeneous binary core \mathbb{A} , the relation $R_3 := R_1 \bowtie R_2$ and O_1, O_2, O_3 some orbitals. Then all of the following hold:

- if R_1 contains an O_1O_2 -tuple t_1 and R_2 contains an $O_2^{-1}O_3$ -tuple t_2 then R_3 contains an O_1O_3 -tuple,
- if t_1 and t_2 are injective, then R_3 contains all injective O_1O_3 -tuples
- if O_1, O_2, O_3 are = and both t_1 and t_2 are non-constant, then R_3 contains all non-constant ==-tuples.

Proof: Consider the first item in the formulation of the observation and the case where R_1 and R_2 are ternary. Let (a_1, a_2, a_3) be an O_1O_2 -tuple in R_1 and (b_2, b_3, b_4) an $O_2^{-1}O_3$ -tuple in R_2 . Since \mathbb{A} is homogeneous, there exists an automorphism $\alpha \in \operatorname{Aut}(\mathbb{A})$ sending (b_3, b_2) to (a_2, a_3) . Let a_4 be $\alpha(b_4)$. Observe that an assignment $\mathbf{a} : \{x_1, x_2, x_3, y\} \to A$ such that $\mathbf{a}(x_1) = a_1$, $\mathbf{a}(x_2) = a_2$, $\mathbf{a}(y) = a_3$ and $\mathbf{a}(x_3) = a_4$. Since (a_3, a_2, a_4) is in the same orbit as (b_2, b_3, b_4) , we have that the assignment \mathbf{a} satisfies all atomic formulae in (1), we have that R_3 has an O_1O_3 -tuple. The proof for quaternary R_1, R_2 is similar with a difference that we consider an O_1O_2 -tuple (a_1, a_2, a_3, a_4) in $R_1, O_2^{-1}O_3$ -tuple (b_3, b_4, b_5, b_6) in R_2 , and an automorphism $\alpha \in \operatorname{Aut}(\mathbb{A})$ sending (b_4, b_3) to (a_3, a_4) .

Now turn to the second item for ternary relation R_1 which contains an O_1O_2 tuple t_1 and R_2 with an $O_2^{-1}O_3$ -tuple t_2 . Since \mathbb{A} is liberal we have that for any orbital $O_{1,4}$ there exists a substructure of \mathbb{A} over four pairwise different elements a_1, a_2, a_3, a_4 such that (a_1, a_2, a_3) is in the same orbit as $t_1, (a_3, a_2, a_4)$ in the same orbit as t_2 and $(a_1, a_4) \in O_{1,4}$. It follows that R_3 has an O_1O_3 tuple t such that $(t[1], t[3]) \in O_{1,4}$. Since $O_{1,4}$ was chosen arbitrarily, we have that R_3 contains all injective tuples. The proof for quaternary relations is again similar. We just look at a substructure of \mathbb{A} over six elements $a_1, a_2, a_3, a_4, a_5, a_6$ such that (a_1, a_2, a_5, a_6) is in the same orbit as t_1 , the tuple (a_6, a_5, a_3, a_4) is in the same orbit as t_2 and such that for all $(i, j) \in \{(1, 3), (1, 4), (2, 3), (2, 4)\}$ we have $(a_i, a_j) \in O_{i,j}$ for some anti-reflexive orbital $O_{i,j}$. An assignment $\mathbf{a} : \{x_1, \ldots, x_4, y_1, y_2\} \to A$ such that $\mathbf{a}(x_i) = a_i$ for $i \in [4]$ and $\mathbf{a}(y_i) = a_{i+4}$ for $i \in [2]$ clearly satisfies both atomic formulae in (2). Since $O_{i,j}$ with $(i, j) \in$ $\{(1, 3), (1, 4), (2, 3), (2, 4)\}$ were chosen arbitrarily, it follows that R_3 contains all injective O_1O_3 -tuples. It completes the proof of the second item.

Since there are no ternary non-constant ==-tuples, the last item concerns quaternary relations only. We proceed as in the proof for the second item. Let t_1 be a non-constant tuple in R_1 and t_2 a non-constant tuple in R_2 . This time we look at a substructure of A induced by pairwise different a_1, a_2, a_3 such that (a_1, a_2) is in the same orbital as $(t_1[2], t_1[3]), (a_2, a_3)$ in the same orbit as $(t_2[2], t_2[3])$, and (a_1, a_3) in some anti-reflexive orbital $O_{1,3}$. The assignment **a** sending x_1, x_2 to a_1, y_1, y_2 to a_2 and x_3, x_4 to a_3 satisfies both atomic formulae in (2) and hence provides an ==-tuple t for R satisfying $(t[2], t[3]) \in O_{1,3}$. Since $O_{1,3}$ was chosen arbitrarily, we have that R_3 contains all non-constant ==-tuples. It completes the proof for the last item in the observation.

An important part of the proof that a pp-defined ternary relation is a critical ternary relation is to show that it contains relations $(C_1(x_1, x_2) \land D_1(x_2, x_3))$ and $(C_2(x_1, x_2) \land D_2(x_2, x_3))$, see Definition 12. To this end we will use the following observation.

Observation 23 Let R_1, R_2 be two ternary relations with fo-definitions in a liberal finitely bounded homogeneous binary core \mathbb{A} such that R_1 contains all injective O_1O_2 -tuples and R_2 contains all injective $O_2^{-1}O_3$ -tuples for some anti-reflexive orbitals O_1, O_2, O_3 then $R_1 \bowtie R_2$ contains the relation

 $O_1(x_1, x_2) \wedge O_3(x_2, x_3).$

Proof: If O_3 is different from O_1^{-1} , then the observation follows by Observation 22. Indeed, in this case it is enough to prove that R_3 contains all injective O_1O_3 -tuples. Otherwise, we repeat the proof of Observation 22 to show that R has all injective $O_1O_1^{-1}$ -tuples but we have also to show that R_3 contains an $O_1O_1^{-1}$ tuple t with t[1] = t[3]. To this end, consider any pairwise different $a_1, a_2, a_3 \in A$ such that $(a_1, a_2) \in O_1, (a_2, a_3) \in O_2$ and $(a_1, a_3) \in O_{1,3}$ for some anti-reflexive orbital $O_{1,3}$. They exist since A is liberal. The relation R_1 contains all injective O_1O_2 -tuples and R_2 contains all injective $O_2^{-1}O_1^{-1}$ -tuples, hence in particular R_1 contains (a_1, a_2, a_3) and R_2 contains (a_3, a_2, a_1) . It follows that the assignment $\mathbf{a} : \{x_1, x_2, x_3, y\}$ sending x_1, x_3 to a_1, x_2 to a_2 and y to a_3 satisfies both atomic formulae in (1) and provides an $O_1O_1^{-1}$ -tuple with t[1] = t[3] in R_3 .

The next observation will be used in a similar context as Observation 23. The difference is that here we will look at quaternary not ternary relations R_1, R_2 and at \bowtie_3 not \bowtie .

Observation 24 Let R_1, R_2 be two quaternary relations with fo-definitions in a liberal finitely bounded homogeneous binary core \mathbb{A} such that R_1 contains all injective O_1O_2 -tuples and R_2 contains all injective $O_2^{-1}O_3$ -tuples or all O_1, O_2, O_3 are = and both R_1, R_2 contain all non-constant ==-tuples then $R_1 \bowtie_3 R_2$ contains:

$$O_1(x_1, x_2) \wedge O_3(x_2, x_3).$$

Proof: Since A is liberal, we have that for anti-reflexive orbitals $O_{1,3}$ it contains a substructure over pairwise different elements a_1, a_2, a_3, a_4, a_5 such that $(a_1, a_2) \in O_1, (a_4, a_5) \in O_2, (a_2, a_3) \in O_3$ and $(a_1, a_3) \in O_{1,3}$. Since R_1 contains all injective O_1, O_2 -tuples and R_2 contains all injective $O_2^{-1}O_3$ -tuples, we have that an assignment $\mathbf{a} : \{x_1, x_2, x_3, y_1, y_2\} \to A$ such that $\mathbf{a}(x_i) = a_i$ for $i \in [3]$ and $\mathbf{a}(y_i) = a_{i+3}$ for $i \in [2]$ satisfies both atomic formulae in (3). Since $O_{1,3}$ may be arbitrary, it proves that R_3 has all injective O_1O_3 -tuples. If $O_3 = O_1^{-1}$, then we also have to show that R_3 has an $O_1O_1^{-1}$ -tuple t with t[1] = t[3]. The proof is similar with a difference that we choose a_1, a_2, a_3, a_4, a_5 with $a_1 = a_3$. \Box

4.2 Some Implications with no Bounded Strict Width

We are now ready to prove that certain ternary and quaternary relations ppdefine a critical ternary relation and hence do not have bounded strict width.

Lemma 25 Let \mathbb{A} be a liberal finitely bounded homogeneous binary core and \mathbb{B} a first-order expansion of \mathbb{A} which pp-defines a ternary relation R that entails no equalities but entails

$$\varphi(x_1, x_2, x_3) \equiv (x_1 = x_2 \lor x_2 = x_3 \lor x_1 = x_3).$$

Then R, and hence \mathbb{B} do not have bounded strict width.

Proof: Since R entails no equalities, we have that if R entails any subformula of φ , then this subformula has at least two disjuncts. Assume without loss of generality that in this case R entails $(x_1 = x_2 \lor x_2 = x_3)$. Since R entails no equalities we have that $(R'(x_1, x_2, x_3) \equiv R(x_1, x_2, x_3) \land x_1 \neq x_3)$ is equivalent to $((C(x_1, x_2) \land x_2 = x_3) \lor (x_1 = x_2 \land D(x_2, x_3)))$ for some anti-reflexive $C, D \subseteq A^2$. By Observation 6, the relation R' is pp-definable in \mathbb{B} .

Further, since \mathbb{A} is liberal, it is easy to see that the relation

$$(R''(x_1, x_2, x_3) \equiv \exists y \ R'(x_1, x_2, y) \land R'(x_3, x_2, y)) \tag{4}$$

equals $(S(x_1, x_2, x_3) \equiv (C_1(x_1, x_2) \land C^{-1}(x_2, x_3)) \lor (x_1 = x_2 \land x_2 = x_3))$. Indeed, any assignment $\mathbf{a} : \{x_1, x_2, x_3, y\} \to A$ satisfying both atomic formulae in (4) either sends (x_1, x_2) to some pair (a_1, a_2) in C_1 or to the same element in A. In the former case, we have that $\mathbf{a}(x_2) = \mathbf{a}(y)$, and hence $(\mathbf{a}(x_2), \mathbf{a}(x_3)) = (a_2, a_3)$ for some $(a_2, a_3) \in C^{-1}$ Since \mathbb{A} is liberal, we may choose (a_1, a_2, a_3) so that (a_1, a_3) is in any orbital $O_{1,3}$ if $(a_1, a_2) \in O_{1,2}$ and (a_2, a_3) in $O_{2,3} = O_{1,2}^{-1}$ and so that $O_{1,3}$ is any anti-reflexive orbital if $O_{2,3} \neq O_{1,2}^{-1}$. It follows that R''contains $(R_1(x_1, x_2, x_3) \equiv (C_1(x_1, x_2) \wedge C^{-1}(x_2, x_3)))$ and that any tuple in R''with the two first coordinates being different is in R_1 . On the other hand, any assignment $\mathbf{a} : \{x_1, x_2, x_3, y\} \to A$ satisfying both atomic formulae in (4) that sends (x_1, x_2) to the same element in A, satisfies also $(\mathbf{a}(x_2), \mathbf{a}(y)) \in D$ and hence $(\mathbf{a}(x_2) = \mathbf{a}(x_3))$. It follows that R'' equals S.

Clearly the relation S efficiently entails both: $(C(x_1, x_2) \implies C^{-1}(x_2, x_3))$ and $(C^{-1}(x_2, x_3) \implies C(x_1, x_2))$ and clearly C and = are pp-definable in \mathbb{B} . Hence R is a critical ternary relation over $(\mathbb{B}, C, =, C^{-1}, =)$. It follows by Proposition 15 that \mathbb{B} do not have bounded strict width.

Assume now that R entails no subformulae of φ . Since R, however, entails φ , we have that also in this case $(R'(x_1, x_2, x_3) \equiv R(x_1, x_2, x_3) \land x_1 \neq x_3)$ is equivalent to $((C(x_1, x_2) \land x_2 = x_3) \lor (x_1 = x_2 \land D(x_2, x_3)))$ for some anti-reflexive $C, D \subseteq A^2$. The proof is hence identical as in the previous case. It completes the proof of the lemma.

As we will see $(\mathbb{B}, C, D, O, =)$ -implications where \mathbb{B} is a first-order expansion of a liberal finitely bounded homogeneous binary core and O an anti-reflexive orbital do not have bounded strict width. We start with ternary implications.

Lemma 26 Let \mathbb{A} be a liberal finitely bounded homogeneous binary core and \mathbb{B} a first-order expansion of \mathbb{A} , let R be a ternary relation pp-definable in \mathbb{B} that entails no equalities and efficiently entails

$$(O(x_1, x_2) \implies x_2 = x_3)$$

for some anti-reflexive orbital O. Then $\mathbb B$ does not have bounded strict-width.

Proof: Since R entails no equalities, by Lemma 25, we have that R contains an injective tuple t such that in particular $(t[1], t[2]) \in P$ for some anti-reflexive orbital P different from O. Consider now the relation:

$$R'(x_1, x_2, x_3) \equiv \exists y \ R(x_1, x_2, y) \land O(x_3, y)$$
(5)

and observe that R^\prime

- efficiently entails $\eta_O := (O(x_1, x_2) \implies O^{-1}(x_2, x_3)),$
- contains an injective OO^{-1} -tuple, and
- an injective PP^{-1} tuple.

The first item follows from the fact that the relation R entails $(O(x_1, x_2) \implies x_2 = x_3))$. For the second item consider a substructure of \mathbb{A} induced by three pairwise different elements a_1, a_2, a_3 such that $(a_1, a_2) \in O$ and $(a_2, a_3) \in O^{-1}$.

Since A is liberal such a_1, a_2, a_3 exist. Observe that (a_1, a_2, a_3) is an OO^{-1} -tuple in R'. Finally, consider a substructure of A over four pairwise different elements (a_1, a_2, a, a_3) such that (a_1, a_2, a) is in the same orbit as t, $(a_2, a_3) \in P^{-1}$, and $(a_3, a) \in O$. Observe that an assignment sending y to a and x_i to a_i for $i \in [3]$ satisfies all atomic formulae in (5). It follows that R contains an injective PP^{-1} -tuple.

Then by Observation 22 and 23, it follows that both the relation $R_3 := ((R' \bowtie R') \bowtie (R' \bowtie R'))$ and the relation $(R'_3(x_1, x_2, x_3) \equiv R_3(x_3, x_2, x_1))$ contain both

- $R_1 := (O(x_1, x_2) \land O^{-1}(x_2, x_3))$ and
- $R_2 := (P(x_1, x_2) \land P^{-1}(x_2, x_3)).$

Clearly both R_3 and $R_3(x_3, x_2, x_1)$ efficiently entails η_O . It follows that the relation $(R_4(x_1, x_2, x_3) \equiv (R_3(x_1, x_2, x_3) \land R'_3(x_1, x_2, x_3)))$ is a critical ternary relation over $(\mathbb{B}, O, P, O^{-1}, P^{-1})$ and the lemma follows by Proposition 15. \Box

Roughly speaking, the following corollary says that if a ternary relation R contains an O_1O_2 tuple then it contains an injective O_1P -tuple for some orbital P provided an orbital O_1 is anti-reflexive and an injective O_1O_2 -tuple provided both orbitals O_1 and O_2 are anti-reflexive.

Corollary 27 Let \mathbb{A} be a liberal finitely bounded homogeneous binary core and \mathbb{B} a first-order expansion of \mathbb{A} with bounded strict width that pp-defines a ternary relation R that entails no equalities. Then for any list of pairwise different twoelement sets of indices $\{i_1, j_1\}, \ldots, \{i_m, j_m\}$ with $m \in [2]$ such that there exists a tuple $t \in R$ satisfying $t[i_k] \neq t[j_k]$ for all $k \in [m]$, the relation R contains an injective tuple t' such that $(t'[i_k], t'[j_k])$ is in the same orbital as $(t[i_k], t[j_k])$ for all $k \in [m]$.

Proof: Assume the contary. Then, since R entails no equalities, in the case where m = 1 the relation R efficiently entails $(O(x_a, x_b) \implies x_c = x_d)$ for some $a, b, c, d \in [3]$. It contradicts Lemma 26 and completes the proof in this case.

For m = 2 we have that R entails the formula $(O(x_a, x_b) \land O(x_b, x_c) \implies x_a = x_c)$ where $\{a, b, c\} = \{1, 2, 3\}$. Without loss of generality assume that R entails $(O(x_1, x_2) \land O(x_1, x_3) \implies x_2 = x_3)$. Observe that R contains a tuple t such that $(t[1], t[2]), (t[1], t[3]) \in O$ and (t[2] = t[3]). Since R entails neither $(O(x_1, x_2) \implies x_2 = x_3)$ nor $(O(x_1, x_3) \implies x_2 = x_3)$. It follows that R contains a tuple t such that $(t[1], t[3]) \in O$ and $(t[2] \neq t[3])$. It follows that $(R'(x_1, x_2, x_3) \equiv R(x_1, x_2, x_3) \land O(x_1, x_3))$ efficiently entails $(O(x_1, x_2) \implies x_2 = x_3)$. The corollary follows by Lemma 26.

We now turn to showing that quaternary $(\mathbb{B}, C, D, O, =)$ -implications with anti-reflexive orbitals O have no bounded strict width. We first consider the case where the implication has an injective tuple.

Lemma 28 Let \mathbb{A} be a liberal finitely bounded homogeneous binary core and \mathbb{B} a first-order expansion of \mathbb{A} that pp-defines a quaternary relation R that efficiently

entails

$$(O(x_1, x_2) \implies x_3 = x_4)$$

and contains an injective tuple. Then $\mathbb B$ does not have bounded strict width.

Proof: By the assumptions of the lemma, the relation R contains an injective tuple t_P such that $(t_P[1], t_P[2]) \in P$ for some anti-reflexive orbital P different from O. By Corollary 27, the projection of the relation R to its three first arguments contains an injective tuple t' with $(t'[1], t'[2]) \in O$, and hence R contains an O=-tuple t_O such that $t_O[1], t_O[2], t_O[3]$ are pairwise different. Further, we claim that the relation

$$R'(x_1, x_2, x_3, x_4) \equiv \exists y \ R(x_1, x_2, x_3, y) \land O(x_4, y)$$
(6)

satisfies all of the following:

- it efficiently entails $(O(x_1, x_2) \implies O^{-1}(x_3, x_4)),$
- contains an injective OO^{-1} -tuple, and
- an injective PP^{-1} -tuple.

The first item follows clearly by the fact that R efficiently entails ($\eta_{\text{Eq}} := (O(x_1, x_2) \implies x_3 = x_4)$). For the second item, consider four pairwise different elements a_1, a_2, a_3, a_4 in A such that (a_1, a_2, a_3, a_3) is in the same orbit as t_O and $(a_3, a_4) \in O^{-1}$. Since \mathbb{A} is liberal the elements a_1, a_2, a_3, a_4 exist. Observe that the assignment $\mathbf{a} : \{x_1, \ldots, x_4, y\} \to A$ sending x_i to a_i for $i \in [4]$ and y to a_3 satisfies all atomic formulae in (6). It follows that R' has an injective OO^{-1} -tuple. For the last item consider pairwise different $a_1, a_2, a_3, a, a_4 \in A$ such that (a_1, a_2, a_3, a) is in the same orbit as t_P and a_4 is such that $(a_4, a) \in O$ and $(a_3, a_4) \in P^{-1}$. Again, a_1, a_2, a_3, a, a_4 exist since \mathbb{A} is liberal. Observe that the assignment $\mathbf{a} : \{x_1, \ldots, x_4, y\} \to A$ sending x_i to a_i for $i \in [4]$ and y to a satisfies all atomic formulae in (6). Hence R' contains an injective PP^{-1} -tuple.

Then by Observation 22 and 24, we have that the ternary relation $R_3 := ((R' \bowtie R') \bowtie_3 (R' \bowtie R'))$ and the relation $(R'_3(x_1, x_2, x_3) \equiv R_3(x_3, x_2, x_1))$ contains both

- $R_1 := (O(x_1, x_2) \land O^{-1}(x_2, x_3))$ and
- $R_2 := (P(x_1, x_2) \land P^{-1}(x_2, x_3)).$

By the definitions of \bowtie and \bowtie_3 , both R_3 and R'_3 entail ($\eta_O \equiv (O(x_1, x_2) \implies O^{-1}(x_2, x_3))$). Since both the relation R_3 and R'_3 contain R_2 , we have that both R_3 and R'_3 efficiently entail η_O . It follows that $(R_3(x_1, x_2, x_3) \land R_3(x_3, x_2, x_1))$ is a critical ternary relation over $(\mathbb{B}, O, P, O^{-1}, P^{-1})$ and the lemma follows by Proposition 15.

Next, we look at $(\mathbb{B}, C, D, O, =)$ -implications R where O is an anti-reflexive orbital and R does not have an injective tuple.

Lemma 29 Let \mathbb{A} be a liberal finitely bounded homogeneous binary core and \mathbb{B} a first-order expansion of \mathbb{A} that pp-defines a quaternary relation R that entails

$$(x_1 = x_2 \lor x_3 = x_4)$$

and entails no equalities. Then $\mathbb B$ does not have bounded strict width.

Proof: Observe that every binary core with one injective orbital only is isomorphic to $(\mathbb{N}; \neq, =)$. By Theorem 1 in [22], see also the introduction in this paper, it follows that every relation R that has both a first-order definition in $(\mathbb{N}; \neq, =)$ and bounded strict width is definable by a conjunction of disjunctions of disequalities, i.e., conjunctions of clauses of the form $(x_1 \neq y_1 \lor \cdots \lor x_k \neq y_k)$. Such relations R clearly do not efficiently entail $(x_1 = x_2 \lor x_3 = x_4)$, and hence we can assume that \mathbb{A} contains at least two different anti-reflexive orbitals. This assumption will be in effect in the remainder of the proof of the lemma.

Let now $C, D \subseteq \neq$ be two anti-reflexive binary relations such that every O=tuple in R with an antireflexive O satisfies $O \in C$ and every =O-tuple in R with an antireflexive O satisfies $O \in D$. We set O to be some orbital in C and P to be some anti-reflexive orbital different from O. We claim that the relation

$$R'(x_1, x_2, x_3, x_4) \equiv \exists y \ R(x_1, x_2, x_3, y) \land O(y, x_4),$$

satisfies all of the following:

- R' efficiently entails $(P(x_3, x_4) \implies x_1 = x_2)$
- R' contains an injective tuple.

For the first item consider an assignment $\mathbf{a} : \{x_1, x_2, x_3, x_4, y\}$ sending (x_3, x_4) to P. Since O is different from P, we have $\mathbf{a}(x_3) \neq \mathbf{a}(y)$. It implies $\mathbf{a}(x_1) = \mathbf{a}(x_2)$. Hence R' entails $(P(x_3, x_4) \implies x_1 = x_2)$. Further, consider (a_1, a_2, a_3, a, a_4) such that $(a_1, a_2, a_3, a) \in R$ and $a_3 \neq a$, $(a_3, a_4) \in P$, and $(a, a_4) \in O$. Since \mathbb{A} is liberal and R entails no equalities, such a tuple exists. It completes the proof of the first item.

For the second item consider (a_1, a_2, a_3, a, a_4) such that $(a_1, a_2, a_3, a) \in R$, $a_3 = a$, and $(a_3, a_4) \in O$, and $(a_1, a_2) \in O'$ for some anti-reflexive orbital O'. We will show that (a_1, a_2, a_3, a_4) is an injective O'O-tuple in R' or Rhas no bounded strict width. If the former is not the case, then R' entails $(O'(x_1, x_2) \land O(x_3, x_4) \implies x_k = x_l)$ for some $k, l \in [4]^2 \setminus \{\{1, 2\}, \{3, 4\}\}$. Since R entails no equalities, $\{k, l\} \cap \{1, 2\} \neq \emptyset$ and $\{k, l\} \cap \{3, 4\} \neq \emptyset$, we have by Lemma 26 that R' either has no bounded strict width and we are done or entails neither $O'(x_1, x_2) \implies x_k = x_l$ nor $O(x_3, x_4) \implies x_k = x_l$. It follows that $(R''(x_1, x_2, x_3, x_4) \land O(x_3, x_4))$ efficiently entails $(O'(x_1, x_2) \implies x_k = x_l)$. It follows by Lemma 26 that R has no bounded strict width and completes the proof of the second item.

Now, it is easy to see that Lemma 28 applied to \mathbb{A} , \mathbb{B} and the relation $(R''(x_1, x_2, x_3, x_4) \equiv R'(x_3, x_4, x_1, x_2))$ completes the proof of this lemma. \Box

Practically, the following corollary says that for any non-constant tuple t of a quaternary relation under consideration one can find an injective tuple t' that agrees with t on the 'injective part'.

Corollary 30 Let \mathbb{A} be a liberal finitely bounded homogeneous binary core and \mathbb{B} a first-order expansion of \mathbb{A} with bounded strict width that pp-defines a quaternary relation R that entails no equalities. Then for any list of pairwise different two-element sets of indices $\{i_1, j_1\}, \ldots, \{i_m, j_m\}$ with $i_k, j_k \in [4]$ for all $k \in [m]$ such that there exists a tuple $t \in R$ satisfying $t[i_k] \neq t[j_k]$ for all $k \in [m]$, the relation contains an injective tuple t' such that $(t'[i_k], t'[j_k])$ is in the same orbital as $(t[i_k], t[j_k])$ for all $k \in [m]$.

Proof: Assume the contrary and let O_{i_k,j_k} with $k \in [m]$ be an orbital of $t[i_k, j_k]$. Then there exists a minimal subset $\mathbb{I} \subseteq \{\{i_1, j_1\}, \dots, \{i_m, j_m\}\}$ such that the formula

$$(R(x_1, x_2, x_3, x_4) \land \bigwedge_{\{i,j\} \in \mathbb{I}} O_{i,j}(x_j, x_j))$$

entails $y_1 = y_2$ for some $y_1, y_2 \in \{x_1, x_2, x_3, x_4\}$. Assume without loss of generality that $(y_1, y_2) = (x_3, x_4)$. Let now $\mathbb{I}' := \mathbb{I} \setminus \{i_0, j_0\}$ for some $\{i_0, j_0\}$ in \mathbb{I} . Since R entails no equalities, the pair $\{i_0, j_0\}$ exists. Observe that the new formula

$$(R'(x_1, x_2, x_3, x_4) \equiv R(x_1, x_2, x_3, x_4) \land \bigwedge_{\{i, j\} \in \mathbb{I}'} O_{i, j}(x_i, x_j))$$

efficiently entails $(O_{i_0,j_0}(x_{i_0}, x_{j_0}) \implies x_3 = x_4)$. If there is such a choice of \mathbb{I}' and i_0, j_0 that $\{i_0, j_0\} \cap \{3, 4\} \neq \emptyset$, then the corollary follows by Lemma 26. Otherwise $\{i_0, j_0\} = \{1, 2\}$ and R' contains a tuple t such that $t_1[1], t_1[2], t_1[3]$ are pairwise different and $t_1[3] = t_1[4]$ as well as a tuple t_2 such that $t_2[2], t_2[3], t_1[4]$ are pairwise different. If we can find t_2 which is injective, then we are done by Lemma 28. Otherwise R' entails $(x_1 = x_2 \lor x_3 = x_4)$. Since R' entails no equalities, the corollary follows by Lemma 29.

4.3 Composing Implications

We will now define the way the 'implications' can be composed. We will be composing the implications originating from a 'cyclic' instance of $CSP(\mathbb{B})$ for implicationally hard \mathbb{B} in order to obtain a critical ternary relation.

Definition 31 Let the relation R_1 be a $(\mathbb{B}, C, D, C_1, D_1, L_1, P_1)$ -implication and R_2 a $(\mathbb{B}, D, F, D_1, F_1, L_2, P_2)$ -implication. We write $R_3 := R_1 \circ R_2$ for a relation obtained in one of the following ways:

• if both R_1 and R_2 are quaternary implications and $P_1 = L_2$, then R_3 is quaternary and $R_3(x_1, x_2, x_3, x_4)$ is defined by:

$$\exists y_1 \exists y_2 \ R_1(x_1, x_2, y_1, y_2) \land R_2(y_1, y_2, x_3, x_4), \tag{7}$$

• if both R_1 and R_2 are quaternary implications and $P_1 \neq L_2$, then R_3 is quaternary and $R_3(x_1, x_2, x_3, x_4)$ is defined by:

$$\exists y_1 \exists y_2 \ R_1(x_1, x_2, y_1, y_2) \land R_2(y_2, y_1, x_3, x_4), \tag{8}$$

• if the relation R_1 is a quaternary (L_1, P_1) -implication and the relation R_2 a ternary (L_2, P_2) -implication such that $P_1 = L_2$, then R_3 is quaternary and $R_3(x_1, x_2, x_3, x_4)$ is defined by

$$\exists y \ R_1(x_1, x_2, y, x_3) \land R_2(y, x_3, x_4) \tag{9}$$

• if the relation R_1 is a quaternary (L_1, P_1) -implication and the relation R_2 a ternary (L_2, P_2) -implication such that $P_1 \neq L_2$, then R_3 is quaternary and $R_3(x_1, x_2, x_3, x_4)$ is defined by

$$\exists y \ R_1(x_1, x_2, x_3, y) \land R_2(y, x_3, x_4) \tag{10}$$

• if the relation R_1 is a ternary (L_1, P_1) -implication and R_2 is a quaternary (L_2, P_2) -implication such that $(P_1 = L_2)$, then R_3 is quaternary and $R_3(x_1, x_2, x_3, x_4)$ is defined by

$$\exists y \ R_1(x_1, x_2, y) \land R_2(x_2, y, x_3, x_4) \tag{11}$$

• if the relation R_1 is a ternary (L_1, P_1) -implication and R_2 is a quaternary (L_2, P_2) -implication such that $(P_1 \neq L_2)$, then R_3 is quaternary and $R_3(x_1, x_2, x_3, x_4)$ is defined by

$$\exists y \ R_1(x_1, x_2, y) \land R_2(y, x_2, x_3, x_4) \tag{12}$$

• if the relation R_1 is a ternary (L_1, P_1) -implication and the relation R_2 a ternary (L_2, P_2) -implication with $P_1 = L_1$, then R_3 is quaternary and $R_3(x_1, x_2, x_3, x_4)$ is defined by

$$R_1(x_1, x_2, x_3) \wedge R_2(x_2, x_3, x_4) \tag{13}$$

• if the relation R_1 is a ternary (L_1, P_1) -implication and the relation R_2 a ternary (L_2, P_2) -implication with $P_1 \neq L_2$, then R_3 is ternary and $R_3(x_1, x_2, x_3, x_4)$ is defined by

$$\exists y \ R_1(x_1, x_2, y) \land R_2(y, x_2, x_3).$$
(14)

We will also write $(R_1)^{\circ k}$ for $(R_1 \circ \cdots \circ R_1)$.

k timesWe will now prove that \circ states for the composition of implications, i.e., that this operation also returns an implication. **Lemma 32** Let the relation R_1 be a $(\mathbb{B}, C, D, C_1, D_1, L_1, P_1)$ -implication and R_2 a $(\mathbb{B}, D, F, D_1, F_1, L_2, P_2)$ -implication such that both have a first-order definition in a liberal finitely bounded homogeneous structure \mathbb{A} . Then the relation $R_3 := R_1 \circ R_2$ is a $(\mathbb{B}, C, F, C_1, F_1, L_1, P_2)$ -implication such that R_3 has an O_1O_3 -tuple for some orbitals O_1 and O_3 if

- either P₁ = L₂, R₁ has an O₁O₂-tuple and R₂ has an O₂O₃-tuple for some orbital O₂, or
- P₁ ≠ L₂, R₁ has an O₁O₂-tuple and R₂ has an O₂⁻¹O₃-tuple for some orbital O₂.

Proof: Consider first the case where both R_1, R_2 are quaternary implications such that $L_2 = P_1$ and let (a_1, a_2, a_3, a_4) and (b_3, b_4, b_5, b_6) be an O_1O_2 -tuple in R_1 and an O_2O_3 -tuple in R_2 for some orbitals O_1, O_2, O_3 , respectively. Since A is homogeneous, we have that there exists an automorphism α of A sending (a_3, a_4) to (b_3, b_4) . Let $(b_1, b_2) = (\alpha(a_1), \alpha(a_2))$. It is now easy to see that an assignment $a(x_1) = b_1$, $a(x_2) = b_2$, $a(y_1) = b_3$, $a(y_2) = b_4$, $a(x_3) = b_5$, $a(x_4) = b_6$ satisfies both atomic formulae in (7) and since $(b_5, b_6) \in O_3$, it provides the desired O_1O_3 tuple in R_3 . Since $\operatorname{proj}_{3,4} R_1 = \operatorname{proj}_{1,2} R_2$, we have that $\operatorname{proj}_{1,2} R_3 = \operatorname{proj}_{1,2} R_1$ and $\operatorname{proj}_{3,4} R_2 = \operatorname{proj}_{3,4} R_3$. Further, $R_1(x_1, x_2, y_1, y_2)$ entails $(C_1(x_i, x_j) \Longrightarrow$ $D_1(y_k, y_l)$ and $R_2(y_1, y_2, x_3, x_4)$ entails $(D_1(y_k, y_l) \implies F_1(x_m, x_n))$ where $i, j, k, l \in [2]$ and $m, n \in \{3, 4\}$. It follows that R_3 entails $(\eta(x_1, x_2, x_3, x_4) \equiv$ $(C_1(x_i, x_j) \implies F_1(x_k, x_l)))$. Since $C_1 \subsetneq \operatorname{proj}_{1,2} R_3 = \operatorname{proj}_{1,2} R_1$ and $F_1 \subsetneq$ $\operatorname{proj}_{3,4} R_3 = \operatorname{proj}_{3,4} R_2$, it follows that η is efficiently entailed by R_3 . In order to complete the proof for this case, it is enough to prove that R_3 entails no equalities. Indeed, since R_1 entails no equalities, it contains an O_1O_2 -tuple where O_1 is anti-reflexive. By Corollary 30, it follows that R_3 contains an injective tuple t' with $(t'[1], t'[2]) \in O_1$. It implies that R_3 implies no equalities, and hence that $R_3 := R_1 \circ R_2$ is a $(\mathbb{B}, C, F, C_1, F_1, L_1, P_2)$ -implication.

The proof in the case where both R_1 and R_2 are quaternary implications but $L_2 \neq P_1$ is similar with a difference that we look at an O_1O_2 -tuple (a_1, a_2, a_3, a_4) and an $O_2^{-1}O_3$ -tuple (b_3, b_4, b_5, b_6) and search for an automorphism α that sends (a_3, a_4) to (b_4, b_3) .

Further, the proof in all other cases in analogous to either of the cases considered above. In the last case where R_3 is a ternary implication, we use Corollary 27 instead of 30.

4.4 Bipartite Digraph of Implications

In this subsection we introduce a bipartite graph that reflects the structure of OP-tuples inside a pair of ternary or quaternary implications. Observe that the relations $(C_i(x_1, x_2) \land D_i(x_2, x_3))$ with $i \in [2]$ definable in liberal finitely bounded homogeneous binary cores, which are important ingredients of critical ternary relations, have all possible OP-tuples such that $O \subseteq C_i$ and $P \subseteq D_i$.

Consider a finitely bounded homogeneous structure \mathbb{A} and $C \subseteq A^2$. We will write $\operatorname{Vert}_L(C)$ and $\operatorname{Vert}_R(C)$ for the set $\{O_L \mid O \text{ is an orbital contained in } C\}$

and the set $\{O_R \mid O \text{ is an orbital contained in } C\}$, respectively. To keep things simple, we say that a $(\mathbb{B}, C, C, C_1, L, P)$ -implication is a $(\mathbb{B}, C, C_1, L, P)$ -implication.

Definition 33 Let the relation R_1 and the relation R_2 be both $(\mathbb{B}, C, C_1, \rightarrow, \leftarrow)$ -implications. We define a bipartite directed graph \mathcal{B}_{R_1,R_2} over two disjoint sets of vertices $\operatorname{Vert}_L(C)$ and $\operatorname{Vert}_R(C)$. The digraph \mathcal{B}_{R_1,R_2} contains

- an arc $(O_L, P_R) \in \operatorname{Vert}_L(C) \times \operatorname{Vert}_R(C)$ if the relation R_1 contains a tuple (a_1, a_2, a_3, a_4) such that $(a_1, a_2) \in O$ and $(a_3, a_4) \in P^{-1}$ and
- an arc $(P_R, O_L) \in \operatorname{Vert}_R(C) \times \operatorname{Vert}_L(C)$ if the relation R_2 contains a tuple (a_1, a_2, a_3, a_4) such that $(a_1, a_2) \in P$ and $(a_3, a_4) \in O^{-1}$.

We say that $\{O_L, P_R\}$ is a symmetric edge in \mathcal{B}_{R_1,R_2} if it contains both an arc (O_L, P_R) and (P_R, O_L) .

We say that a subset S of the vertices of a digraph is a strongly connected component if it is a maximal set of vertices such that for any two vertices $u, v \in S$ there is a path connecting u and v. A set of vertices S is a set of strongly connected components if it can be partitioned into S_1, \ldots, S_k so that every S_i with $i \in [k]$ is a strongly connected component. We say that a (set of) strongly connected components is a *sink* if every arc originating in S finishes in S and that S is a *source* if every arc finishing in S also originates in S.

Lemma 34 Let R_1, R_2 be $(\mathbb{B}, C, C_1, \rightarrow, \leftarrow)$ -implications, \mathbb{B} a first-order expansion of a liberal finitely bounded homogeneous binary core \mathbb{A} . Then $(\operatorname{Vert}_L(C_1) \cup \operatorname{Vert}_R(C_1))$ is a set of strongly connected components which is a sink in \mathcal{B}_{R_1,R_2} .

Moreover, there exist non-empty $D_1, F_1 \subseteq C \setminus C_1$ such that $(\operatorname{Vert}_L(D_1) \cup \operatorname{Vert}_R(F_1))$ is a source in \mathcal{B}_{R_1,R_2} .

Proof: For the first part of the lemma observe that every arc originating in some vertex in $(\operatorname{Vert}_L(C_1) \cup \operatorname{Vert}_R(C_1))$ finishes in a vertex in $(\operatorname{Vert}_L(C_1) \cup \operatorname{Vert}_R(C_1))$. Indeed, it follows by the fact that both R_1, R_2 are $(\mathbb{B}, C, C_1, \rightarrow, \leftarrow)$ -implications.

Observe that the second part of the lemma follows by the facts that the component $(\operatorname{Vert}_L(C_1) \cup \operatorname{Vert}_R(C_1))$ is a sink in \mathcal{B}_{R_1,R_2} , the fact that $C_1 \subsetneq C$ and that \mathcal{B}_{R_1,R_2} is finite and smooth, i.e., has no sources and no sinks. The graph \mathcal{B}_{R_1,R_2} is smooth since $\operatorname{proj}_{1,2} R_1 = \operatorname{proj}_{1,2} R_2 = \operatorname{proj}_{4,3} R_1 = \operatorname{proj}_{4,3} R_2$. \Box

The following lemma may be simply proved by induction using Lemma 32.

Lemma 35 Let R_1, R_2 be $(\mathbb{B}, C, C_1, \rightarrow, \leftarrow)$ -implications, \mathbb{B} a first-order expansion of a liberal finitely bounded homogeneous binary core \mathbb{A} and O_L, P_R a pair of vertices in \mathcal{B}_{R_1,R_2} such that there is a path from O_L to P_R of length 2k + 1 for some $k \in \mathbb{N}$. Then $(R_1 \circ R_2)^{\circ k} \circ R_1$ is a $(\mathbb{B}, C, C_1, \rightarrow, \leftarrow)$ -implication and has an OP^{-1} -tuple.

We will say that the relation R is a complete $(\mathbb{B}, C, C_1, \rightarrow, \leftarrow)$ -implication if every strongly connected component in $\mathcal{B}_{R,R}$ satisfies both of the following:

- it is of the form $\operatorname{Vert}_L(D_1) \cup \operatorname{Vert}_R(D_1)$ for some $D_1 \subseteq C$,
- every strongly connected component of $\mathcal{B}_{R,R}$ is a complete bipartite digraph.

Example. Let \mathbb{B} be a first-order expansion of a liberal finitely bounded homogeneous binary core and R a critical ternary relation over $(\mathbb{B}, C_1, D_1, C_1^{-1}, D_1^{-1})$ given by

$$(C_1(x_1, x_2) \land C_1^{-1}(x_2, x_3)) \lor (D_1(x_1, x_2) \land D_1^{-1}(x_2, x_3)).$$

Observe that the relation R is a complete $(\mathbb{B}, C_1 \cup D_1, C_1, \rightarrow, \leftarrow)$ -implication.

We finally prove that we can always define a complete implication given two appropriate implications.

Lemma 36 Let R_1, R_2 be both $(\mathbb{B}, C, C_1, \rightarrow, \leftarrow)$ -implications where \mathbb{B} is a first-order expansion of a liberal finitely bounded homogeneous binary core \mathbb{A} . Then they pp-define a complete $(\mathbb{B}, C, C_1, \rightarrow, \leftarrow)$ -implication.

Proof: We say that two vertices in a bipartite digraph are loosely connected if they are in the same strongly connected component of the digraph and the shortest cycle they are both involved in is of length strictly greater than 2.

Consider now two $(\mathbb{B}, C, C_1, \rightarrow, \leftarrow)$ -implications R_1, R_2 such that not every strongly connected component of \mathcal{B}_{R_1,R_2} is complete. Then there are loosely connected vertices O_L, P_R in \mathcal{B}_{R_1,R_1} such that there is an arc (P_R, O_L) in \mathcal{B}_{R_1,R_2} but the shortest path from O_L to P_R is of length 2k + 1 with $k \geq 1$. By Lemma 35, we have that $R_3 := (R_1 \circ R_2)^{\circ k} \circ R_1$ contains an OP^{-1} -tuple as well as all $O'P'^{-1}$ -tuples such that $\{O', P'\}$ is a symmetric edge in \mathcal{B}_{R_1,R_2} . Since R_3 is also a $(\mathbb{B}, C, C_1, \rightarrow, \leftarrow)$ -implication, we have that the number of loosely connected vertices in \mathcal{B}_{R_3,R_2} is strictly less than in \mathcal{B}_{R_1,R_2} . Since the graphs under consideration are finite, it is enough to repeat the whole procedure a finite number of times in order to obtain a pair of implications R', R'' without losely connected vertices.

In order to complete the proof of the lemma observe that $R' \circ R''$ is the desired complete $(\mathbb{B}, C, C_1, \rightarrow, \leftarrow)$ -implication.

5 Implicationally Simple Languages

In this section we characterize the relational width of a large class of constraint languages which we call implicationally simple. We start with a precise definition of $\mathcal{G}_{\mathcal{I}}$.

Graph of a CSP-instance. Let \mathcal{I} be a $(2, \mathbb{L}_{\mathbb{A}})$ -minimal instance of $\text{CSP}(\mathbb{B})$ over variables \mathcal{V} of which we assume that \mathcal{I} entails no equalities. We define the implication graph $\mathcal{G}_{\mathcal{I}}$ of \mathcal{I} to be a directed graph over vertices which are triples of the form $((v_1, v_2), C)$ where $v_1, v_2 \in \mathcal{V}$ and $C \subsetneq \mathcal{I}_{v_1, v_2}$ is a binary non-empty relation pp-definable in \mathbb{B} . There is an arc in one of the two following situations:

- an arc $((x_1, x_2), C_1), ((x_2, x_3), D_1))$ with pairwise different $x_1, x_2, x_3 \in \mathcal{V}$ if there exists a constraint $\mathbf{C} \in \mathcal{I}$ whose scope contains $\{x_1, x_2, x_3\}$,
 - $\operatorname{proj}_{x_1, x_2, x_3} \mathbf{C} = ((x_1, x_2, x_3), R')$ and
 - R' is a ternary $(\mathbb{B}, \mathcal{I}_{x_1, x_2}, C_1, \mathcal{I}_{x_2, x_3}, D_1)$ -implication,
- an arc $((x_1, x_2), C_1), ((x_3, x_4), D_1))$ with paiwise different $x_1, x_2, x_3, x_4 \in \mathcal{V}$ if there exists a constraint $\mathbf{C} \in \mathcal{I}$ whose scope contains $\{x_1, x_2, x_3, x_4\}$ and

$$-\operatorname{proj}_{x_1,x_2,x_3,x_4} \mathbf{C} = ((x_1, x_2, x_3, x_4), R')$$
 and

- the relation R' is a quaternary $(\mathbb{B}, \mathcal{I}_{x_1, x_2}, C_1, \mathcal{I}_{x_3, x_4}, D_1)$ -implication.

We are now ready to define a new class of languages.

Definition 37 Let \mathbb{B} be a first-order expansion of a finitely bounded homogeneous binary core \mathbb{A} . We say that a structure \mathbb{B} is implicationally simple if for every $(2, \mathbb{L}_{\mathbb{A}})$ instance \mathcal{I} of $CSP(\mathbb{B})$ the graph $\mathcal{G}_{\mathcal{I}}$ is acyclic.

If \mathbb{B} is not implicationally simple, then we say that it is implicationally hard.

Naturally, every acyclic $\mathcal{G}_{\mathcal{I}}$ contains a sink which is a singleton $\{((v_1, v_2), C\})\)$ for some $v_1, v_2 \in \mathcal{V}$ and $C \subsetneq A^2$. We will now show that in this case an instance $\mathcal{I}[(v_1, v_2) := C]$ obtained from \mathcal{I} by narrowing down the projection of every constraint in \mathcal{I} containing (v_1, v_2) in its scope to C has the good properties of \mathcal{I} .

Observation 38 Let \mathcal{I} be a non-trivial $(2, \mathbb{L}_{\mathbb{A}})$ -minimal instance of $CSP(\mathbb{B})$ where \mathbb{B} is a first-order expansion of a finitely bounded homogeneous binary core \mathbb{A} such that $\{((v_1, v_2), C)\}$ is a sink in $\mathcal{G}_{\mathcal{I}}$. Then the instance $\mathcal{I}[(v_1, v_2) := C]$ is also a non-trivial $(2, \mathbb{L}_{\mathbb{A}})$ -minimal instance of $CSP(\mathbb{B})$.

Proof: Since \mathcal{I} is $(2, \mathbb{L}_{\mathbb{A}})$ -minimal, we have that $\mathcal{I}[(v_1, v_2) := C]$ is nontrivial. To see that $\mathcal{I}[(v_1, v_2) := C]$ is $(2, \mathbb{L}_{\mathbb{A}})$ -minimal consider any constraint $\mathbf{C} = ((x_1, \ldots, x_r), R)$ in \mathcal{I} and any pair of variables $v_3, v_4 \subseteq \{x_1, \ldots, x_r\}$. We have to show that for every orbital $O \subseteq \mathcal{I}_{v_3, v_4}$ there is an assignment $\mathbf{a} : \{x_1, \ldots, x_r\} \to A$ satisfying $R(x_1, \ldots, x_r)$ such that $(\mathbf{a}(v_3), \mathbf{a}(v_4)) \in O$. If $\{v_1, v_2\} \not\subseteq \{x_1, \ldots, x_r\}$, then we are done by the fact that \mathcal{I} is $(2, \mathbb{L}_{\mathbb{A}})$ minimal. In the other case, since $\{((v_1, v_2), C)\}$ is a sink in $\mathcal{G}_{\mathcal{I}}$, the projection of $R(x_1, \ldots, x_r)$ to $\{v_1, v_2, v_3, v_4\}$ does not efficiently entail $(C(v_1, v_2) \to D(v_3, v_4))$ for any $D \subsetneq I_{v_3, v_4}$. Hence there exists an assignment $\mathbf{a} : \{x_1, \ldots, x_r\} \to A$ satisfying $R(x_1, \ldots, x_r)$ such that $(\mathbf{a}(v_3), \mathbf{a}(v_4)) \in O$ and $(\mathbf{a}(v_1), \mathbf{a}(v_2)) \in C$. It completes the proof of the observation. \Box The above observation is the key to showing that every non-trivial $(2, \mathbb{L}_{\mathbb{A}})$ minimal instance of $CSP(\mathbb{B})$ with implicationally simple \mathbb{B} has a solution.

Theorem 39 Let \mathbb{A} be a finitely-bounded homogenous binary core and \mathbb{B} a firstorder expansion of \mathbb{A} which is implicationally simple. Then every non-trivial $(2, \mathbb{L}_{\mathbb{A}})$ -minimal instance has a solution.

Proof: Indeed, either there exist $v_1, v_2 \in \mathcal{V}$ such that I_{v_1,v_2} consists of at least two orbitals or all I_{v_1,v_2} consist of exactly one orbital O. In the former case, since all binary relations pp-definable in \mathbb{B} contain an orbital, the graph $\mathcal{G}_{\mathcal{I}}$ contains at least one vertex $((v_1, v_2), C)$ for $v_1, v_2 \in \mathcal{V}$ such that \mathcal{I}_{v_1,v_2} consists of at least two orbits. Since \mathbb{B} is implicationally simple, the graph $\mathcal{G}_{\mathcal{I}}$ is acyclic. In particular, it has a sink $\{((v_1, v_2), C)\}$ for some C. Hence, we can use Observation 38 and simplify the considered instance by replacing \mathcal{I} with $\mathcal{I}[(v_1, v_2) := C]$. The new instance is also non-trivial and $(2, \mathbb{L}_{\mathbb{A}})$ -minimal. The process of simplifying the instance terminates when every \mathcal{I}_{v_1,v_2} with $v_1, v_2 \in \mathcal{V}$ consists of one orbit only. Clearly, every solution to this simplified instance is a solution to the original one.

From now we assume that \mathcal{I} is $(2, \mathbb{L}_{\mathbb{A}})$ -minimal, $\mathcal{V} = \{v_1, \ldots, v_n\}$ and for all $i, j \in [n]$ we have that \mathcal{I}_{v_i, v_j} consists of exactly one orbital. Consider a structure Δ over the signature τ of \mathbb{A} whose elements are variables in \mathcal{V} and $(v_i, v_j) \in \mathbb{R}^{\Delta}$ with $\mathbb{R} \in \tau$ if and only if $\mathcal{I}_{v_i, v_j} = \mathbb{R}^{\mathbb{A}}$. Observe first that E equal to $=^{\Delta} \cup \{(v_i, v_i) \mid i \in [n]\}$ is an equivalence relation. It is clearly reflexive and symmetric. Suppose E is not transitive. Then there are $v_i, v_j, v_k \in V$ such that $(v_i, v_j) \in =^{\Delta}$ and $(v_j, v_k) \in =^{\Delta}$ but $v_i, v_k \notin =^{\Delta}$. It implies that a constraint \mathbb{C} whose scope contains $\{v_i, v_j, v_k\}$, which is in \mathcal{I} by $(2, \mathbb{L}_{\mathbb{A}})$ -minimality, has an empty relation. It contradicts the fact that \mathcal{I} is non-trivial.

In order to complete the proof of the theorem we will show that Δ/E admits an embedding to \mathbb{A} . Assume it is not the case. Then there exists a finite structure $\Gamma \in \mathcal{F}_{\mathbb{A}}$ that embeds into \mathbb{A} . Since the size of Γ is at most $\mathbb{L}_{\mathbb{A}}$, it contradicts the fact that \mathcal{I} is $(2, \mathbb{L}_{\mathbb{A}})$ -minimal and non-trivial. It follows that Δ/E embeds and Δ homomorphically maps into \mathbb{A} . It follows that \mathcal{I} has a solution.

6 FO-Expansions of Liberal Finitely Bounded Homogeneous Binary Cores without Bounded Strict Width

In this section we show that if a first-order expansion of a liberal finitely bounded homogeneous binary core is implicationally hard, then it has no bounded strict width. As a result, we obtain that every first-order expansion of a liberal finitely bounded homogeneous binary core with bounded strict width is implicationally simple and hence its relational width is characterized by Theorem 39. **Lemma 40** Let \mathbb{A} be a liberal finitely bounded homogeneous binary core and \mathbb{B} a first-order expansion of \mathbb{A} which is implicationally hard. Then \mathbb{B} pp-defines

- a quaternary $(\mathbb{B}, C, C_1, \rightarrow, \leftarrow)$ -implication, or
- a ternary $(\mathbb{B}, C, C_1, \rightarrow, \leftarrow)$ -implication.

Proof: Since \mathbb{B} is implicationally hard, there exists an instance \mathcal{I} of CSP(\mathbb{B}) such that $\mathcal{G}_{\mathcal{I}}$ contains a path $((v, z), C_1) = ((v_1, z_1), D_1), \ldots, ((v_n, z_n), D_n) = ((v, z), C_1)$ such that for all $j \in [n - 1]$ the instance \mathcal{I} contains a constraint \mathbf{C} whose scope contains $v_j, z_j, v_{j+1}, z_{j+1}$ and one of the following holds:

- either the elements $x_1 = v_j, x_2 = z_j = v_{j+1}, x_3 = z_{j+1}$ are pairwise different, $\operatorname{proj}_{x_1,x_2,x_3} \mathbf{C} = ((x_1, x_2, x_3), R'_j)$ and the relation R'_j is a ternary $(\mathbb{B}, \mathcal{I}_{v_j,z_j}, \mathcal{I}_{v_{j+1},z_{j+1}}, C_j, C_{j+1})$ -implication, or
- the elements $x_1 = v_j, x_2 = z_j, x_3 = v_{j+1}, x_4 = z_{j+1}$ are pairwise different, $\operatorname{proj}_{x_1, x_2, x_3, x_4} \mathbf{C} = ((x_1, x_2, x_3, x_4), R'_j)$ and R'_j is a quaternary $(\mathbb{B}, \mathcal{I}_{v_j, z_j}, \mathcal{I}_{v_{j+1}, z_{j+1}}, C_j, C_{j+1})$ -implication.

It follows that $R' := ((R'_1 \circ R'_2) \circ \ldots \circ R'_k)$ is well defined and by Lemma 32, the relation R' is a $(\mathbb{B}, \mathcal{I}_{v,z}, \mathcal{I}_{v,z}, C_1, C_1)$ -implication. Assume first that R' is ternary. If the relation R' is a ternary $(\mathbb{B}, \mathcal{I}_{v,z}, C_1, \leftarrow, \rightarrow)$ -implication, then it is also a ternary $(\mathbb{B}, (\mathcal{I}_{v,z})^{-1}, (C_1)^{-1}, \rightarrow, \leftarrow)$ -implication and then the lemma follows. Otherwise, R' is a $(\rightarrow, \rightarrow)$ -implication or a (\leftarrow, \leftarrow) -implication, and then, by Lemma 40, the relation $R' \circ R'$ is a quaternary $(\mathbb{B}, \mathcal{I}_{v,z}, C_1)$ -implication. From now on we may therefore assume that R' is quaternary.

If the relation R' is a $(\rightarrow, \leftarrow)$ -implication, then we are done. Otherwise R' is an (L, P)-quaternary implication with either $L \neq \rightarrow$ or $P \neq \leftarrow$. But then

- $R'(x_2, x_1, x_3, x_4)$ is a $(\rightarrow, \leftarrow)$ -implication in the case where $L = \leftarrow$ and $R = \leftarrow$,
- $R'(x_1, x_2, x_4, x_3)$ is a $(\rightarrow, \leftarrow)$ -implication in the case where $L = \rightarrow$ and $R = \rightarrow$, and
- $R'(x_2, x_1, x_4, x_3)$ is a $(\rightarrow, \leftarrow)$ -implication in the case where $L = \leftarrow$ and $R = \rightarrow$.

It completes the proof of the lemma.

We are now in the position to show that first-order expansions of liberal finitely bounded homogenous binary cores \mathbb{A} with bounded strict width are implicationally simple and hence their relational width is $(2, \mathbb{L}_{\mathbb{A}})$.

Lemma 41 Let \mathbb{A} be a liberal finitely bounded homogeneous binary core and \mathbb{B} a first-order expansion of \mathbb{A} which is implicationally hard. Then \mathbb{B} has no bounded strict width.

Proof: By Lemma 40, we may assume that \mathbb{B} pp-defines an $(\mathbb{B}, C', C'_1, \rightarrow$ $, \leftarrow$)-implication R'. By Lemma 36 that R' is a complete implication. By Lemma 34, we have that $(\mathbf{Vert}_L(C'_1) \cup \mathbf{Vert}_R(C'_1))$ is a set of strongly connected components which is a sink in $\mathcal{B}_{R',R'}$. Let $C''_1 \subseteq A^2$ be such that $(\operatorname{Vert}_L(C''_1) \cup$ $\operatorname{Vert}_R(C_1'')$ form a strongly connected component included in $(\operatorname{Vert}_L(C_1') \cup$ $\operatorname{Vert}_R(C'_1)$ which is a sink in $\mathcal{B}_{R',R'}$ and $D''_1 \subseteq A^2$ such that $(\operatorname{Vert}_L(D''_1) \cup$ $\operatorname{Vert}_R(D_1'')$ is a strongly connected component which is a source in $\mathcal{B}_{R',R'}$. Since the component $(\operatorname{Vert}_L(C''_1) \cup \operatorname{Vert}_R(C''_1))$ is a sink, we have that R' is a $(\mathbb{B}, C', C''_1, \rightarrow, \leftarrow)$ -implication, Now, if either C'_1 or D'_1 is neither anti-reflexive nor it is =, then we replace R' with $R'' := R'(x_1, x_2, x_3) \land x_1 \neq x_2 \land x_2 \neq x_3$ x_3 if R' is ternary and with $R'' := R'(x_1, x_2, x_3, x_4) \land x_1 \neq x_2 \land x_3 \neq x_4$ if R' is quaternary. Set now $C = C' \setminus \{=\}, C_1 = C''_1 \cap C$ and $D_1 = D''_1 \cap C$ C and observe that $\mathcal{B}_{R'',R''}$ is a digraph induced in $\mathcal{B}_{R',R'}$ by $(\mathbf{Vert}_L(C) \cup$ $\operatorname{Vert}_R(C)$). It follows that R'' is a complete $(\mathbb{B}, C, C_1, \rightarrow, \leftarrow)$ -implication where the strongly connected component $(\operatorname{Vert}_L(C_1) \cup \operatorname{Vert}_R(C_1))$ is a sink and the strongly connected component ($\operatorname{Vert}_L(D_1) \cup \operatorname{Vert}_R(D_1)$) is a source in $\mathcal{B}_{R'',R''}$.

From now on we will assume that the relation R is a complete $(\mathbb{B}, C, C_1, \rightarrow$, \leftarrow)-implication where C_1 is either = or it is anti-reflexive and $(\mathbf{Vert}_L(C_1) \cup$ $\operatorname{Vert}_R(C_1)$ is a strongly connected component which is a sink in $\mathcal{B}_{R,R}$ as well as that $D_1 \subseteq C \setminus C_1$ is either = or it is anti-reflexive and $(\operatorname{Vert}_L(D_1) \cup \operatorname{Vert}_R(D_1))$ is a strongly connected component which is a source in $\mathcal{B}_{R,R}$. Clearly, either C_1 or D_1 is anti-reflexive. We will now prove that R pp-defines a ternary $(\mathbb{B}, C, C_1, \rightarrow, \leftarrow)$ -implication R_1 that contains for all $F_1 \subseteq \{C_1, D_1\}$ the relation $(F_1(x_1, x_2) \land F_1^{-1}(x_2, x_3))$. In the case where R is ternary we set $R_1 := (R \bowtie$ $R \bowtie (R \bowtie R)$ which is equivalent to setting $R_1 := (R \circ R) \circ (R \circ R)$. Hence, by Lemma 32, the relation R is a ternary $(\mathbb{B}, C, C_1, \rightarrow, \leftarrow)$ -implication. Further, if F_1 is =, then clearly R_1 has a ternary ==-tuple. If F_1 is anti-reflexive, then consider any orbitals O_1, O_3 in F_1 . Since R is a complete implication, by Corollary 27, it contains an injective $O_1 O_2^{-1}$ -tuple as well as an injective $O_2O_3^{-1}$ -tuple for some orbital $O_2 \subseteq F_1$, and hence by Observations 22 and 23, we have that R_1 contains $(O_1(x_1, x_2) \land O_3^{-1}(x_2, x_3))$, and in consequence that R_1 contains $(F_1(x_1, x_2) \land F_1^{-1}(x_2, x_3))$. Clearly, also the relation $((R_2(x_1, x_2, x_3) \equiv$ $R_1(x_1, x_2, x_3) \cap R_1(x_3, x_2, x_1)))$ contains $(F_1(x_1, x_2) \wedge F_1^{-1}(x_2, x_3))$ for $F_1 \subseteq$ $\{C_1, D_1\}$. Further, observe that \mathcal{B}_{R_2,R_2} has only symmetric edges. Hence, since $(\operatorname{Vert}_L(D_1) \cup \operatorname{Vert}_R(D_1))$ is a sink in $\mathcal{B}_{R,R}$ and also in \mathcal{B}_{R_2,R_2} , we have that $D_1(x_1, x_2) \equiv R_2(x_1, x_2, x_3) \wedge O^{-1}(x_2, x_3)$ for some orbital contained in D_1 , and hence D_1 is pp-definable in \mathbb{B} . It follows that R_2 is either a critical ternary relation over $(\mathbb{B}, C_1, D_1, C_1^{-1}, D_1^{-1})$ if C_1 is anti-reflexive or a critical ternary relation over $(\mathbb{B}, D_1, C_1, D_1^{-1}, C_1^{-1})$ if D_1 is anti-reflexive. By appeal to Proposition 15, the structure \mathbb{B} does not have bounded strict width.

We now turn to the case where R is a complete quaternary $(\mathbb{B}, C, C_1, \rightarrow, \leftarrow)$ implication. This time we look at $R_1 := ((R \bowtie R) \bowtie_3 (R \bowtie R))$ and this time we will first show that for all $F_1 \in \{C_1, D_1\}$, R_1 contains the relation $R_F := F_1(x_1, x_2) \wedge F_1^{-1}(x_3, x_4)$. If F_1 is = and R contains a constant tuple, then R_1 also contains a constant tuple and hence R_F . If F_1 is = and R_1 contains a non-constant ==-tuple, then by Observations 22 and 24, R_1 contains an ==-tuple and hence R_F . For the case where F_1 is anti-reflexive we again consider any orbitals O_1, O_3^{-1} in F. Since R is a complete implication, it contains an injective $O_1O_2^{-1}$ -tuple as well as an injective $O_2O_3^{-1}$ -tuple for some some orbital $O_2 \subseteq F_1$. It follows by Observations 22 and 24 that R_1 contains R_F also in the case where F_1 is anti-reflexive. Consider now $(R_2(x_1, x_2, x_3) \equiv$ $R_1(x_1, x_2, x_3) \cap R_1(x_3, x_2, x_1))$ and observe that \mathcal{B}_{R_2,R_2} has only symmetric edges. Since $(\operatorname{Vert}_L(D_1) \cup \operatorname{Vert}_R(D_1))$ is a strongly connected component in \mathcal{B}_{R_2,R_2} we can pp-define D_1 . Hence R_2 is either a critical ternary relation over $(\mathbb{B}, C_1, D_1, C_1^{-1}, D_1^{-1})$ if C_1 is anti-reflexive or a critical ternary relation over $(\mathbb{B}, D_1, C_1, D_1^{-1}, C_1^{-1})$ if D_1 is anti-reflexive. By Proposition 15, we have that \mathbb{B} does not have bounded strict width. \Box

We are now ready to provide the proof for Theorem 1.

Proof of Theorem 1 The result follows by Lemma 41 and Theorem 39.

7 Future Work

In this paper we characterized the relational width of first order expansions of liberal finitely bounded homogenous binary cores with bounded strict width. First of all it is natural to ask what happens if the binary core is not liberal. By Proposition 3 we have that the method in this paper does not work when we forbid structures of size 3. On the other hand, by the result in [23] it holds that the relational width of first-order expansions of homogenous graphs \mathbb{A} even with $\mathbb{L}_{\mathbb{A}} = 3$ is still $(2, \mathbb{L}_{\mathbb{A}})$. Could it be true for all binary cores \mathbb{A} ?

Last but not least, what happens to the relational width when we allow first-order expansions of cores \mathbb{A} of arbitrary arity k? We believe that with an appropriate notion of liberal k-ary cores \mathbb{A} one can show that their first-order expansions \mathbb{B} with bounded strict width have relational width $(k, \mathbb{L}_{\mathbb{A}})$. Could it be true for all reducts of finitely bounded homogeneous structures?

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