# On The Relational Width of First-Order Expansions of Finitely Bounded Homogeneous Binary Cores with Bounded Strict Width* 

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#### Abstract

The relational width of a finite structure, if bounded, is always $(1,1)$ or $(2,3)$. In this paper we study the relational width of first-order expansions of finitely bounded homogeneous binary cores where binary cores are structures with equality and some anti-reflexive binary relations such that for any two different elements $a, b$ in the domain there is exactly one binary relation $R$ with $(a, b) \in R$.

Our main result is that first-order expansions of liberal finitely bounded homogeneous binary cores with bounded strict width have relational width (2, MaxBound) where MaxBound is the size of the largest forbidden substructure, but is not less than 3, and liberal stands for structures that do not forbid certain finite structures of small size. This result is built on a new approach and concerns a broad class of structures including reducts of homogeneous digraphs for which the CSP complexity classification has not yet been obtained.


## 1 Introduction

An instance of the constraint satisfaction problem (CSP) consists of a number of variables and a number of local restrictions on variables called constraints. The question is whether there exists a global assignment to variables that satisfies all constraints. The CSP naturally generalizes SAT, expresses a number of other natural problems including $k$-coloring, solving equations over finite fields but, at least among theoreticians, is associated mainly to the question on dichotomy [17], i.e., is every right-hand side restriction $\operatorname{CSP}(\mathbb{B})$ of the CSP in P or

[^0]NP-complete? A relational structure $\mathbb{B}$, known also as a (constraint) language or a template, restricts the available constraints to these that can be modelled by relations in $\mathbb{B}$. It is known already for a while that the dichotomy for $\operatorname{CSP}(\mathbb{B})$ over finite structures exists. Indeed, the Feder Vardi conjecture, on the existence of the dichotomy, has been confirmed by Zhuk [24] and independently by Bulatov [13].

The problem is that already a very simple scheduling problem with precedence constraints of the form $(X<Y)$ cannot be properly expressed as $\operatorname{CSP}(\mathbb{B})$ if the domain of $\mathbb{B}$ is finite and scheduling problems are what practitioners think of when they hear of the CSP. In order to express a richer class of problems including many scheduling problems as well as problems in spatial and temporal reasoning [7] one considers $\omega$-categorical structures $\mathbb{B}$ that, although infinite, share many nice properties with finite structures. In particular they admit a simple finite representation and the algebraic approach to the complexity of CSP which stands behind the both dichotomy proofs is also applicable in this context. (The precise definition of $\omega$-categoricity as well as many other well-known notions used in the introduction are defined formally in the remainder of the paper, usually in Section2, ) It is no dichotomy to look for among all $\omega$-categorical CSPs (CSPs over $\omega$-categorical templates) [6]. Thus, one considers a subclass - (first-order) reducts of (countably infinite) finitely bounded homogeneous structures for which a dichotomy is conjectured. In what follows we will call this conjecture the infinite dichotomy conjecture. In contrast to $\omega$-categorical CSPs, all CSPs over reducts of finitely bounded homogeneous structures are in NP and more importantly, the infinite dichotomy conjecture is backed by an algebraic dichotomy [3] delineating algebras corresponding to structures with no non-trivial symmetries and such algebras with non-trivial symmetries. It is already known that algebras with no non-trivial symmetries correspond to NPcomplete problems. The infinite tractability conjecture states that a $\operatorname{CSP}(\mathbb{B})$ is in P (tractable) always when the corresponding algebra contains some non-trivial symmetries (polymorphisms). A similar tractability conjecture concerning finite algebras was confirmed by Bulatov and Zhuk.

While in the finite case, the algorithm solving tractable $\operatorname{CSP}(\mathbb{B})$ is based on two prevailing general algorithmic techniques: local-consistency methods [2] and the 'few subpowers' algorithm 19, the development of general algorithmic techniques and establishing the limits of their applicability in the infinite case are rather in their infancy. Two important exceptions are: an algebraic characterization of $\omega$-categorical structures with bounded strict width 4, 5] and the lifiting theorem [10] which lifts the tractability from finite CSP. Since it is already known that the tractability of some tractable reducts of $(\mathbb{Q} ;<)$ a.k.a. temporal languages cannot be explained by the lifiting theorem, the development of general algoritmic techniques for the CSPs within the scope of the infinite tractability conjecture and understanding the limits of their applicability seems inevitable. Natural research questions in this context concern local-consistency methods. Firstly, because the algebraic characterization of finite structures whose CSP may be solved in this way is considered to be an important step towards establishing the dichotomy. Secondly, because local consistency methods
are ubiquitous in constraint solving, for instance, in the context of qualitative calculi in spatial and temporal reasoning [21]. In this paper we consider one of these natural questions.

A structure $\mathbb{B}$ has bounded width if $\operatorname{CSP}(\mathbb{B})$ is solvable by the local-consistency algorithm. Equivalently, $\operatorname{CSP}(\mathbb{B})$ is solvable by an algorithm establishing $(k, l)$-minimiality for some natural numbers $k \leq l$. In this case we say that $\mathbb{B}$ has relational width $(k, l)$ and if such $k, l$ exist that $\mathbb{B}$ has bounded relational width [1, 12]. The relational width of $\mathbb{B}$ proved to be a natural way to measure the amount of consistency needed to solve $\operatorname{CSP}(\mathbb{B})$. In particular, it is known that if the relational width of a finite structure is bounded, then it is $(1,1)$ or $(2,3)$ [1]. This characterization is based on the algebraical characterization of finite structures with bounded (relational) width, and since its counterpart for reducts of finitely bounded homogeneous structures does not exist, it could be very hard to answer the question of what is the relational width of these structures. But as we already mentioned, there is such a characterization for structures with bounded strict width. A structure $\mathbb{B}$ has strict width $k$ if every partial solution to every $(k, l)$-minimal instance of $\operatorname{CSP}(\mathbb{B})$ can be extended to a total solution. This notion is not only of theoretical interest [17] but also under the name local-to-global consistency has been studied in constraint solving in spatial and temporal reasoning, see e.g. [16].

In this paper we characterize the relational width of first-order expansions of liberal finitely bounded homogeneous binary cores with bounded strict width. All the definitions that are necessary to understand the result are given in the following subsection.

### 1.1 Results

We say that a structure $\mathbb{A}$ over a relational signature (here assumed to be finite) is homogeneous if every local isomorphism between finite substructures of $\mathbb{A}$ may be extended to an automorphism of $\mathbb{A}$. A structure $\mathbb{A}$ over a signature $\tau$ is finitely bounded if there exists a finite set of finite $\tau$-structures $\mathcal{F}_{\mathbb{A}}$ such that a finite structure $\Delta$ embeds into $\mathbb{A}$ if and only if there is no $\Gamma$ in $\mathcal{F}_{\mathbb{A}}$ that embeds into $\Delta$.

All homogeneous graphs, classified in [20, and many homogenenous digraphs, classified in [15], are finitely bounded. In particular, it is known [18] that for any countable set of pairwise non-embedabble tournaments $\mathcal{F}$ there exists a homogeneous digraph $\mathbb{A}$ such that $\mathcal{F}=\mathcal{F}_{\mathbb{A}}$. Such homogeneous digraphs are known as Henson digraphs. In this paper we see homogeneous graphs and homogeneous digraphs over an extended signature and study these structures and many other as binary cores defined in what follows.

Binary Cores. We say that a structure $\mathbb{A}$ over domain $A$ is a binary core if its signature besides $=$ contains only binary anti-reflexive relations $R_{1}, \ldots, R_{\kappa}$ such that for any two different elements $a, b \in A$ there is exactly one $R_{i}$ with $i \in[\kappa]$ such that $(a, b) \in R_{i}$.

Examples. A perfect example of a finitely bounded homogeneous binary core is a homogeneous graph seen over the signature $\{E, N,=\}$ where $N$ contains all different pairs of elements which are not connected by an edge $E$, i.e., $(N(x, y) \equiv$ $(\neg E(x, y) \wedge x \neq y))$. Other examples are homogeneous digraphs seen over the signature $\{\curvearrowright, N,=\}$ where $\curvearrowright$ stands for an arc and $N$ is a non-arc relation, i.e., $(N(x, y) \equiv \neg \curvearrowright(x, y) \wedge \neg \curvearrowright(y, x) \wedge x \neq y)$.

Liberal Structures. We restrict ourselves to these binary cores $\mathbb{A}$ that are additionally liberal, i.e., $\mathcal{F}_{\mathbb{A}}$ contains no finite structures of size $3,4,5$, or 6 . In particular any Henson digraph that forbids tournaments of size 7 or greater only, or a random graph $\mathcal{G}$ seen over the extended signature is a liberal binary core. Indeed, all structures in $\mathcal{F}_{\mathcal{G}}$ are of size less than 3 .

The Main Result. We write $\mathbb{L}_{\mathbb{A}}$ to denote the maximum of 3 and the size of the largest structure in $\mathcal{F}_{\mathbb{A}}$. A first-order expansion $\mathbb{B}$ of $\mathbb{A}$ is an expansion of $\mathbb{A}$ such that all relations in $\mathbb{B}$ have first-order definitions in $\mathbb{A}$. We are now in the position to formulate the main result of this paper.

Theorem 1 Let $\mathbb{A}$ be a liberal finitely bounded homogeneous binary core and $\mathbb{B}$ a first-order expansion of $\mathbb{A}$ with bounded strict width. Then $\mathbb{B}$ has relational width $\left(2, \mathbb{L}_{\mathbb{A}}\right)$.

Examples of first-order expansions of the random graph with bounded strict width were given in 23.

Proposition $2([23])$ Let the structure $\mathbb{B}$ be a first-order expansion of the structure $(A ; E, N,=)$ where $(A ; E)$ is the random graph such that every relation in $\mathbb{B}$ is pp-definable as a conjunction of clauses of the form:

$$
\begin{array}{r}
\left(x_{1} \neq y_{1} \vee \cdots \vee x_{k} \neq y_{1} \vee R\left(y_{1}, y_{2}\right) \vee y_{2} \neq z_{1} \vee\right. \\
\left.\vee \cdots \vee y_{2} \neq z_{l}\right),
\end{array}
$$

where $R \in\{E, N\}$. Then $\mathbb{A}$ has bounded strict width.
Further, by Lemma 8 in [8] and Theorem 1 in [22], we have that an equality language, i.e., a first order expansion of $(\mathbb{N} ;=, \neq)$ has bounded strict width if and only if all the extra relations are pp-definable by a conjunction of disjunctions of disequalities, i.e., clauses of the form:

$$
\left(x_{1} \neq y_{1} \vee \cdots \vee x_{k} \neq y_{k}\right)
$$

In order to prove Theorem 1 we show that all instances of $\operatorname{CSP}(\mathbb{B})$ under consideration are simple in a particular sense. Roughly speaking, for a $\left(2, \mathbb{L}_{\mathbb{A}}\right)$ minimal instance $\mathcal{I}$ of $\operatorname{CSP}(\mathbb{B})$ we construct a digraph $\mathcal{G}_{\mathcal{I}}$ over pairs $((v, x), C)$ where $v, x$ are variables in the instance $\mathcal{I}$ and $C \subsetneq A^{2}$ is a binary relation primitively-positively definable in $\mathbb{B}$. (Primitive positive (pp-)definitions are provided by primitive-positive (pp-)formulas which are first-order formulas built
out of the conjunction, existential quantifiers and atomic formulae only.) There is an arc from $\left(\left(v_{1}, x_{1}\right), C\right)$ to $\left(\left(v_{2}, x_{2}\right), D\right)$ in $\mathcal{G}_{\mathcal{I}}$ if there is a constraint over a relation $R$ whose scope $\left(y_{1}, \ldots, y_{k}\right)$ contains $v_{1}, x_{1}, v_{2}, x_{2}$ and $R\left(y_{1}, \ldots, y_{k}\right)$ entails $\left(C\left(v_{1}, x_{1}\right) \Longrightarrow D\left(v_{2}, x_{2}\right)\right)$ so that $C$ and $D$ are not the whole projections of $R\left(y_{1}, \ldots, y_{k}\right)$ to $\left(v_{1}, x_{1}\right)$ and $\left(v_{2}, x_{2}\right)$, respectively. We say that $\mathbb{B}$ is implicationally simple if every $\mathcal{G}_{\mathcal{I}}$ of every $\left(2, \mathbb{L}_{\mathbb{A}}\right)$-minimal instance $\mathcal{I}$ of $\operatorname{CSP}(\mathbb{B})$ is acyclic. We show that

- all implicationally simple first-order expansions of finitely bounded homogeneous binary cores $\mathbb{A}$ have relational width $\left(2, \mathbb{L}_{\mathbb{A}}\right)$ (in Section 5) and
- all implicationally hard first-order expansions of liberal finitely bounded homogeneous binary cores have no bounded strict width (in Sections 34 and (6).

We would like to mention that not all first-order expansions of finitely bounded homogeneous binary cores with bounded strict width are implicationally simple. In [23] one can find the following.

Proposition $3([23])$ The structure $\mathbb{B}=(A ; E, N,=, R)$ where $(A ; E)$ is $C_{2}^{\omega}$ (two disjoint infinite cliques) and

$$
R\left(x_{1}, x_{2}, x_{3}\right) \equiv\left(\left(E\left(x_{1}, x_{2}\right) \wedge N\left(x_{2}, x_{3}\right)\right) \vee\left(N\left(x_{1}, x_{2}\right) \wedge E\left(x_{2}, x_{3}\right)\right)\right)
$$

has bounded strict width.
Clearly, $\mathbb{B}$ is not implicationally simple. Already $\mathcal{G}_{\mathcal{I}}$ for an instance $\mathcal{I}$ with one constraint over relation $R$ is not acyclic.

### 1.2 Related Work

In [23], the following was proved.
Theorem 4 ([23]) Let $\mathbb{A}=(A ; E, N,=)$ be such that $(A ; E)$ is a homogeneous graph and $\mathbb{B}$ a first-order expansion of $\mathbb{A}$ with bounded strict width. Then $\mathbb{B}$ has relational width $\left(2, \mathbb{L}_{\mathbb{A}}\right)$.

Clearly, Theorem 1 generalizes Theorem 4 when $\mathbb{A}$ is liberal but more importantly, the proof in [23] is based on the complexity classification of CSPs over first-order reducts of homogeneous graphs in [9], while the proof in this paper is based only on the assumptions that

- $\mathbb{A}$ is a liberal finitely bounded homogeneous binary core and that
- $\mathbb{B}$ has bounded strict width.

In particular our result concerns all but finitely many finitely bounded Henson digraphs and many other structures for which CSP classifications has not been provided.

## 2 Preliminaries

We write $[n]$ for $\{1, \ldots, n\}$ and when $t$ is an $n$-tuple we write $t[i]$ with $i \in[n]$ to denote the $i$-th value in $t$.

### 2.1 Structures under consideration

We consider here (countably infinite) finitely bounded homogeneous relational structures over domain $A$ usually denoted by $\mathbb{A}$ and their first-order expansions $\mathbb{B}$ also over domain $A$. In particular we look at first-order expansions $\mathbb{B}$ of (liberal) finitely bounded homogeneous binary cores $\mathbb{A}$. Recall that liberal means that $\mathcal{F}_{\mathbb{A}}$ contains no structures of size $3,4,5$, and 6 . For the sake of simplicity we write $R$ to denote both a relational symbol in a signature of $\mathbb{B}$ as well as the actual relation $R^{\mathbb{B}}$.

We will write $\operatorname{proj}_{i_{1}, \ldots, i_{k}} R$ for an $n$-ary relation $R$ and $\left\{i_{1}, \ldots, i_{k}\right\} \subseteq[n]$ to denote the relation $R^{\prime}\left(y_{1}, \ldots, y_{k}\right)$ defined by the formula

$$
\exists x_{1} \cdots \exists x_{n} R\left(x_{1}, \ldots, x_{n}\right) \wedge \bigwedge_{j \in[k]} y_{j}=x_{i_{j}} .
$$

We write $\operatorname{Aut}(\mathbb{A})$ to denote the set of automorphisms of $\mathbb{A}$. An orbit of a tuple $t$ with values in $A$ wrt. $\operatorname{Aut}(\mathbb{A})$ is the set $\{(\alpha(t[1]), \ldots, \alpha(t[n])) \mid \alpha \in \operatorname{Aut}(A)\}$. When $\mathbb{A}$ is known from the context we simply say an orbit of a tuple instead of an orbit of a tuple wrt. $\operatorname{Aut}(\mathbb{A})$. We also say that $O$ is an orbit if it is an orbit of some tuple. An orbital is an orbit of a tuple with two values.

All the structures under consideration are $\omega$-categorical, i.e., their first-order theories have one countable model up to isomorphism. By the theorem proved independently by Ryll-Nardzewski, Engeler and Svenonius, a structure $\mathbb{B}$ is $\omega$ categorical if and only if its automorphism group is oligomorphic, i.e., for every $n$ the number of orbits of $n$-tuples is finite. Since $\mathbb{A}$ is homogeneous, we have the following.

Observation 5 Let $\mathbb{A}=\left(A ; R_{1}, \ldots, R_{\kappa},=\right)$ be a binary core. Then $R_{i}$ for all $i \in[\kappa]$ is an orbital.

Proof: The equality is clearly an orbital. Since $\mathbb{A}$ is homogeneous we have that there is an automorphism $\alpha \in \operatorname{Aut}(\mathbb{A})$ such that $\left(\alpha\left(a_{1}\right), \alpha\left(a_{2}\right)\right)=\left(a_{3}, a_{4}\right)$ whenever $\left(a_{1}, a_{2}\right),\left(a_{3}, a_{4}\right) \in R_{i}$ and $i \in[\kappa]$. It follows that every $R_{i}$ is a subset of some orbital wrt. $\operatorname{Aut}(\mathbb{A})$. On the other hand, automorphism do not send $\left(a_{1}, a_{2}\right) \in R_{i}$ to $\left(a_{3}, a_{4}\right) \in R_{j}$ for $i \neq j$. The observation follows.

Further, since all structures under consideration have quantifier elimination, i.e., all first-order definable relations are definable without quantifiers, we have that every binary relation fo-definable in a binary core $\mathbb{A}$ is a union of orbitals $R_{i}$ with $i \in[\kappa]$. For a binary relation $C \subseteq A^{2}$ we will write $C^{-1}$ to denote $\left(C^{-1}(x, y) \equiv C(y, x)\right)$. We say that a binary relation $C$ is anti-reflexive if it is contained in $\neq$ or in other words if for all $(a, b) \in C$ we have that $a \neq b$.

It happens that all liberal finitely bounded homogeneous binary cores ppdefine $\neq$.

Observation 6 Let $\mathbb{A}$ be a liberal finitely bounded homogeneous binary core. Then $\mathbb{A}$ pp-defines $\neq$.

Proof: If $\neq$ is an orbital wrt $\operatorname{Aut}(\mathbb{A})$, then we are done. Otherwise, there are at least two different orbitals $O_{1}, O_{2} \subseteq \neq$. We claim that ( $x_{1} \neq x_{2}$ ) is pp-defined by:

$$
\left(\psi\left(x_{1}, x_{2}\right) \equiv\left(\exists x_{0} O_{1}\left(x_{0}, x_{1}\right) \wedge O_{2}\left(x_{0}, x_{2}\right)\right)\right)
$$

Indeed, since $O_{1}$ and $O_{2}$ are different we have that an assignment a : $\left\{x_{1}, x_{2}\right\} \rightarrow$ $A$ such that $\mathrm{a}\left(x_{1}\right)=\mathrm{a}\left(x_{2}\right)$ does not satisfy $\psi$. On the other hand, since $\mathbb{A}$ is liberal, for all anti-reflexive orbitals $O_{3}$ there exist element $a_{0}, a_{1}, a_{2} \in A$ such that $\left(a_{0}, a_{1}\right) \in O_{1},\left(a_{0}, a_{2}\right) \in O_{2}$ and $\left(a_{1}, a_{2}\right) \in O_{3}$. It implies that an assignment a : $\left\{x_{1}, x_{2}\right\} \rightarrow A$ such that $\mathrm{a}\left(x_{i}\right)=a_{i}$ for $i \in[2]$ satisfies $\psi$. Since $O_{3}$ was chosen arbitrarily, we have that $\psi$ is a definition of $\neq$.

In the paper, ternary and quaternary (4-ary) relations are of special interest. We will say that a tuple $t=(t[1], t[2], t[3])$ and $t=(t[1], t[2], t[3], t[4])$ are $O P-$ tuples for some orbitals $O, P$ if $(t[1], t[2]) \in O,(t[2], t[3]) \in P$ and $(t[1], t[2]) \in O$, $(t[3], t[4]) \in P$, respectively.

Further, we will say that a tuple is constant if all its values are the same and that is non-constant otherwise. A tuple is injective if all its values are pairwise different.

### 2.2 Entailment

A first-order formula $\varphi\left(x_{1}, \ldots, x_{n}\right)$ entails a first-order formula $\psi\left(x_{1}, \ldots, x_{n}\right)$ if the formula

$$
\left(\forall x_{1} \cdots \forall x_{n}\left(\varphi\left(x_{1}, \ldots, x_{n}\right) \Longrightarrow \psi\left(x_{1}, \ldots, x_{n}\right)\right)\right)
$$

is valid. Further, we will say that an $n$-ary relation $R$ entails a formula $\psi$ over variables $\left\{x_{1}, \ldots, x_{n}\right\}$ if $R\left(x_{1}, \ldots, x_{n}\right)$ entails $\psi\left(x_{1}, \ldots, x_{n}\right)$.

We also say that an $n$-ary relation $R$ entails no equalities if there are no different $i, j \in[n]$ such that $R$ entails $\left(x_{i}=x_{j}\right)$.

### 2.3 CSP

A constraint $\mathbf{C}$ is a pair $\left(\left(x_{1}, \ldots, x_{k}\right), R\right)$ where $\left(x_{1}, \ldots, x_{k}\right)$ is the $k$-tuple of variables called also the scope of the constraint and $R$ is a $k$-ary relation. We will write $\operatorname{proj}_{x_{i_{1}}, \ldots, x_{i_{l}}} \mathbf{C}$ for the projection of a costraint $\mathbf{C}:=\left(\left(x_{1}, \ldots, x_{k}\right), R\right)$ to a tuple of variables $\left(x_{i_{1}}, \ldots, x_{i_{l}}\right)$ with $\left\{i_{1}, \ldots, i_{l}\right\} \subseteq[k]$. We will have that $\operatorname{proj}_{x_{i_{1}}, \ldots, x_{i_{l}}} \mathbf{C}$ is the constraint $\left(\left(x_{i_{1}}, \ldots, x_{i_{l}}\right), R^{\prime}\right)$ where $R^{\prime}=\operatorname{proj}_{i_{1}, \ldots, i_{l}} R$.

We study the problem $\operatorname{CSP}(\mathbb{B})$ parametrized by first-order expansions of finitely bounded homogeneous structures. The instance $\mathcal{I}$ of $\operatorname{CSP}(\mathbb{B})$ is a set of constraints $\left(\left(x_{1}, \ldots, x_{k}\right), R\right)$ such that $R$ is a relation in $\mathbb{B}$. We say that $\mathcal{I}$
is over variables $\mathcal{V}$ if for every constraint $\left(\left(x_{1}, \ldots, x_{k}\right), R\right)$ in $\mathcal{I}$ we have that $x_{1}, \ldots, x_{k} \in \mathcal{V}$. The question in the problem $\operatorname{CSP}(\mathbb{B})$ is whether there exists a solution to $\mathcal{I}$, i.e., an assignment $\mathbf{s}: \mathcal{V} \rightarrow A$ to variables in $\mathcal{I}$ such that for all constraints $\left(\left(x_{1}, \ldots, x_{k}\right), R\right)$ we have $\left(\mathbf{s}\left(x_{1}\right), \ldots, \mathbf{s}\left(x_{k}\right)\right) \in R$.

Let $W \subseteq \mathcal{V}$. An assignment $\mathbf{a}: W \rightarrow A$ is a partial solution to $\mathcal{I}$ if a satisfies all projections of constraints in $\mathcal{I}$ to variables in $W$. We will sat that $\mathcal{I}$ entails no equalities if no relations in the constraints of $\mathcal{I}$ do.

### 2.4 The universal-algebraic approach

We say that an operation $f: A^{n} \rightarrow A$ is a polymorphism of an $m$-ary relation $R$ iff for all $m$-tuples $t_{1}, \ldots, t_{n} \in R$, it holds that the tuple $\left(f\left(t_{1}[1], \ldots, t_{n}[1]\right), \ldots\right.$, $\left.f\left(t_{1}[m], \ldots, t_{n}[m]\right)\right)$ is also in $R$. We will write $f\left(t_{1}, \ldots, t_{n}\right)$ as a shorthand for the expression $\left(f\left(t_{1}[1], \ldots, t_{n}[1]\right), \ldots, f\left(t_{1}[m], \ldots, t_{n}[m]\right)\right)$. An operation $f$ is a polymorphism of $\mathbb{A}$ if it is a polymorphism of every relation in $\mathbb{A}$. If $f$ : $A^{n} \rightarrow A$ is a polymorphism of $\mathbb{A}, R$, we say that $f$ preserves $\mathbb{A}, R$. A set of polymorphisms of an $\omega$-categorical structure $\mathbb{A}$ forms an algebraic object called an oligomorphic locally closed clone [4, which in particular contains an oligomorphic permutation group [14].

Recall that a first-order formula is a primitive-positive formula (pp-formula) if it is built out of conjunction, existential quantifiers and atomic formulae only. There is a deep connection between the polymorphisms of a structure and the relations pp-definable in that structure.

Theorem 7 ([11]) Let $\mathbb{A}$ be a countable $\omega$-categorical structure. Then $R$ is preserved by the polymorphisms of $\mathbb{A}$ if and only if it has a primitive-positive definition in $\mathbb{A}$.

We say that a set of operations $F$ generates a set of operations $G$ if every $g \in G$ is in the smallest locally-closed clone containing $F$. An operation $f$ of an oligomorphic clone $F$ is called oligopotent if $\{g\}$ where $g(x):=f(x, \ldots, x)$ is generated by the permutations in $F$. We say that a $k$-ary operation $f$ over domain $A$ is a quasi near-unanimity operation (qnu-operation) if

$$
\begin{array}{r}
f(y, x, \ldots, x)=f(x, y, x, \ldots, x)=\cdots= \\
\cdots=f(x, \ldots, x, y)=f(x, \ldots, x)
\end{array}
$$

for all $x, y \in A$.

### 2.5 Widths and Minimality

We will now give a formal defnition of a $(k, l)$-minimal instance.
Definition 8 We say that an instance $\mathcal{I}$ over $\mathcal{V}$ of $\operatorname{CSP}(\mathbb{B})$ is $(k, l)$-minimal with $k \leq l$ if both of the following hold:

- every subset of at most $l$ variables in $\mathcal{V}$ is contained in a scope of some constraint in $\mathcal{I}$ and
- for every at most $k$-element subset of variables $X=\left\{x_{1}, \ldots, x_{k}\right\} \subseteq \mathcal{V}$ and any two constraints $C_{1}, C_{2} \in \mathcal{I}$ whose scopes contain $X$ the projections $\operatorname{proj}_{x_{1}, \ldots, x_{k}} C_{1}$ and $\operatorname{proj}_{x_{1}, \ldots, x_{k}} C_{2}$ are the same.

We say that an instance $\mathcal{I}$ of the CSP is non-trivial if it does not contain a constraint $\left(\left(x_{1}, \ldots, x_{k}\right), \emptyset\right)$. Otherwise, $\mathcal{I}$ is trivial.

Set $k \leq l$. Clearly not every instance $\mathcal{I}$ over variables $\mathcal{V}$ of $\operatorname{CSP}(\mathbb{B})$ for $\mathbb{B}$ over domain $A$ is $(k, l)$-minimal, however, the algorithm that obtains an equivalent $(k, l)$-minimal instance is straightforward and works in time $O\left(|\mathcal{V}|^{m}\right)$ where $m$ is the maximum of $l$ and the largest arity in the signature of $\mathbb{B}$. Indeed, it is enough to introduce a new constraint $\left(\left(x_{1}, \ldots, x_{l}\right), A^{l}\right)$ for all pairwise different variables $x_{1}, \ldots, x_{l} \in \mathcal{V}$ to satisfies the first condition. Then the algorithm removes tuples (orbits) from the relations in constraints in the instance as long as the second condition is not satisfied. It is widely known and easy to prove that an instance $\mathcal{I}^{\prime}$ of the CSP obtained by the described algorithm is equivalent, i.e., has the same set of solution, to the orginal instance $\mathcal{I}$. In particular we have that if $\mathcal{I}^{\prime}$ is trivial, then $\mathcal{I}$ has no solutions. Under a natural assumption that $\mathbb{B}$ contains all at most $l$-ary relations $p p$-definable in $\mathbb{B}$, we have that $\mathcal{I}^{\prime}$ is an instance of $\operatorname{CSP}(\mathbb{B})$. From now on this assumption will be in effect.

Definition 9 A relational structure $\mathbb{B}$ has relational width $(k, l)$ if every $(k, l)$ minimal instance $\mathcal{I}$ of $\mathbb{B}$ has a solution iff it is non-trivial.

A relational structure $\mathbb{B}$ has bounded relational width if it has relational width $(k, l)$ for some natural numbers $k \leq l$.

In this paper we mainly look at $\left(2, \mathbb{L}_{\mathbb{A}}\right)$-minimal instances $\mathcal{I}$ of $\operatorname{CSP}(\mathbb{B})$ for first-order expansions $\mathbb{B}$ of finitely bounded homogeneous binary cores $\mathbb{A}$. For such instances $\mathcal{I}$ and $x, y \in \mathcal{V}$ we will write $\mathcal{I}_{x, y}$ to denote the projection of any constraint in $\mathcal{I}$ to the variables $(x, y)$.
We now turn to strict width.
Definition 10 We say that $\mathbb{B}$ has strict width $k$ if there exists $l$ such that every partial solution of every $(k, l)$-minimal instance of $\operatorname{CSP}(\mathbb{B})$ may be extended to a total solution.

The following theorem provides a characterization of bounded strict width that we use intensively in this paper.

Theorem 11 [5, 4] Let $\mathbb{B}$ be an $\omega$-categorical language. Then the following are equivalent.

1. $\mathbb{B}$ has strict width $k$.
2. $\mathbb{B}$ has an oligopotent $(k+1)$-ary quasi near-unanimity operation as a polymorphism.

Observe that in order to show that some structure $\mathbb{B}$ has no bounded strict width it is enough to show that that they are not preserved by any oligopotent qnu-operations or, by Theorem 7 to pp-define a structure $\mathbb{B}^{\prime}$ of which we already know that has no bounded strict width.

## 3 Critical Ternary Relations

We define a family of structures which is the main source of 'infinite' strict width, i.e., whenever we want to show that some structure does not have bounded strict width we pp-define a critical ternary relation.

Definition 12 We say that a relation $R$ is a critical ternary relation over $\left(\mathbb{B}, C_{1}, C_{2}, D_{1}, D_{2}\right)$ if all of the following hold:

- $R, C_{1}, C_{2}, D_{1}, D_{2}$ are pp-definable in $\mathbb{B}$,
- $C_{1}$ and $C_{2}$ are disjoint and contained in proj $_{1,2} R$
- $D_{1}$ and $D_{2}$ are disjoint and contained in $\operatorname{proj}_{2,3} R$,
- both $C_{1}, D_{1}$ are anti-reflexive,
- either both $C_{2}, D_{2}$ are anti-reflexive or both are $=$,
- $R$ entails $\left(C_{1}\left(x_{1}, x_{2}\right) \Longrightarrow D_{1}\left(x_{2}, x_{3}\right)\right)$
- $R$ entails $\left(D_{1}\left(x_{2}, x_{3}\right) \Longrightarrow C_{1}\left(x_{1}, x_{2}\right)\right)$
- $R$ contains both

$$
\left(R_{1}\left(x_{1}, x_{2}, x_{3}\right) \equiv\left(C_{1}\left(x_{1}, x_{2}\right) \wedge D_{1}\left(x_{2}, x_{3}\right)\right)\right)
$$

and

$$
\left(R_{2}\left(x_{1}, x_{2}, x_{3}\right) \equiv\left(C_{2}\left(x_{1}, x_{2}\right) \wedge D_{2}\left(x_{2}, x_{3}\right)\right)\right)
$$

Before we turn to the main result of this subsection we need two observations.
Observation 13 Let $\mathbb{A}$ be a liberal finitely bounded homogeneous binary core and $R$ a critical ternary relation over $\left(\mathbb{B}, C_{1}, C_{2}, D_{1}, D_{2}\right)$ for some first-order expansion $\mathbb{B}$ of $\mathbb{A}$. Let $k \in \mathbb{N}$ and $I, J \subseteq[k] \backslash\{m\}$ for some $m \in[k]$ be disjoint subsets of indices such that $I \cup J \cup\{m\}=[k]$ and $w, u \in A^{k}$ such that all of the following hold:

- $(w[i], u[i]) \in C_{1}$ for all $i \in I$,
- $(w[m], u[m]) \in C_{2}$, and
- $(w[i], u[i]) \in C_{2}$ for all $i \in J$.

Then there exists $v \in A^{k}$ such that all of the following hold:

- $(w[i], u[i], v[i]) \in R_{1}$ for all $i \in I$,
- $(w[m], u[m], v[m]) \in\left\{\left(x_{1}, x_{2}, x_{3}\right) \in A^{3} \mid C_{2}\left(x_{1}, x_{2}\right) \wedge D_{1}\left(x_{2}, x_{3}\right) \wedge x_{1} \neq\right.$ $\left.x_{3}\right\}$, and
- $(w[i], u[i], v[i]) \in R_{2}$ for all $i \in J$.

Proof: Starting with a substructure of the structure $\mathbb{A}$ induced by the elements $w[1], \ldots, w[k], u[1], \ldots, u[k]$ satisfying the conditions in the formulation of the lemma we will move around the structure $\mathbb{A}$ to find $v[1], \ldots, v[k]$ so that the structure induced by all elements in $w, u$ and $v$ satisfy all the requirements. Assume we are done for $v[1], \ldots, v[i]$ for some $i \in[k]$ and consider $v[i+1]$. Then one of three requirements has to be satisfied by $v[i+1]$. The first case to consider is where $(i+1) \in I$ and we require $(w[i+1], u[i+1], v[i+1]) \in R_{1}$. Since $\mathbb{A}$ is liberal, there are three pairwise different elements $a_{1}, a_{2}, a_{3}$ in $\mathbb{A}$ such that $\left(a_{1}, a_{2}\right)$ are in the same orbital as $(w[i+1], u[i+1])$ and $\left(a_{2}, a_{3}\right)$ is in some (actually in any) orbital contained in $D_{1}$. Since $\mathbb{A}$ is homogeneous there is an automorphism $\alpha$ sending $\left(a_{1}, a_{2}\right)$ to $(w[i+1], u[i+1])$. Then we take $v[i+1]$ to be $\alpha\left(a_{3}\right)$. When $i+1=m$ then either $D_{2}$ is anti-reflexive and we proceed similarly or $D_{2}$ is $=$ and then we set $v[i+1]$ to $u[i+1]$. If $i \in J$, then we proceed as in one of the cases above. Indeed, either $C_{2}, D_{2} \subseteq \neq$ or both $C_{2}, D_{2}$ equal $=$.

We also need to consider a situation from the previous observation where $u, v$ are given and one looks for $w$.

Observation 14 Let $\mathbb{A}$ be a liberal finitely bounded homogeneous binary core and $R$ a critical ternary relation over $\left(\mathbb{B}, C_{1}, C_{2}, D_{1}, D_{2}\right)$ for some first-order expansion $\mathbb{B}$ of $\mathbb{A}$. Let $k \in \mathbb{N}$ and $I, J \subseteq[k] \backslash\{m\}$ for some $m \in[k]$ be disjoint subsets of indices such that $I \cup J \cup\{m\}=[k]$ and $u, v \in A^{k}$ such that

- $(u[i], v[i]) \in D_{1}$ for all $j \in I$,
- $(u[m], v[m]) \in D_{2}$, and
- $(u[i], v[i]) \in D_{2}$ for all $i \in J$.

Then there exists $w \in A^{k}$ such that all of the following hold:

- $(w[j], u[j], v[j]) \in R_{1}$ for all $j \in I$,
- $(w[m], u[m], v[m]) \in\left\{\left(x_{1}, x_{2}, x_{3}\right) \in A^{3} \mid C_{1}\left(x_{1}, x_{2}\right) \wedge D_{2}\left(x_{2}, x_{3}\right) \wedge x_{1} \neq x_{3}\right\}$, and
- $(w[j], u[j], v[j]) \in R_{2}$ for all $j \in J$.

We are now in the position to prove that critical ternary relations ppdefinable in first-order expansions of liberal finitely bounded homogeneous cores have no bounded strict width.

Proposition 15 Let $\mathbb{A}$ be a liberal finitely bounded homogeneous binary core and $R$ a critical ternary relation over $\left(\mathbb{B}, C_{1}, C_{2}, D_{1}, D_{2}\right)$ for some first-order expansion $\mathbb{B}$ of $\mathbb{A}$. Then both $\mathbb{B}$ and $R$ do not have bounded strict width.

Proof: By Theorem 11, it is enough to prove that $R$ is not preserved by any oligopotent qnu-operation. Suppose $\mathbb{B}$ is preserved by a $k$-ary oligopotent quasi near-unanimity operation $f$. The essence of the proof of the proposition is in the following observation.

Let $C$ be a binary relation. We will say that two $k$-tuples $t_{1}, t_{2}$ are $C$ connected on a coordinate $i \in[k]$ if $\left(t_{1}[i], t_{2}[i]\right) \in C$.

Observation 16 For all $i \in\{0, \ldots, k\}$ we have that both of the following hold.

1. For all $(w, u) \in\left(A^{k}\right)^{2}$ such that $u$ is constant, $(w, u)$ are $C_{1}$-connected on $(k-i)$ coordinates and $C_{2}$-connected on $i$ coordinates, we have that $(f(w), f(u)) \in C_{1}$.
2. For all $(u, v) \in\left(A^{k}\right)^{2}$ such that $u$ is constant, $(u, v)$ are $D_{1}$-connected on $(k-i)$ coordinates and $D_{2}$-connected on $i$ coordinates, we have that $(f(u), f(v)) \in D_{1}$.

Proof: The proof goes by the induction on $i$. In the base case where $i=0$, the claim follows by the fact that $C_{1}, D_{1}$ are pp-definable in $\mathbb{B}$.

For the induction step, suppose first that Item 2 fails for some $i>1$ and $J$ with $|J|=i$, i.e., there exist $(u, v) \in\left(A^{k}\right)^{2}$ such that $u$ is constant, $(u, v)$ are $D_{1}$-connected on $(k \backslash[J])$ and $D_{2}$-connected on $J$ and $(f(u), f(v)) \notin D_{1}$. Let $m \in J$. We set $J^{\prime}=J \backslash\{m\}$ and $I^{\prime}=[k] \backslash J$. It follows that $[k]$ is a disjoint union of $I^{\prime},\{m\}$ and $J^{\prime}$ and we have all of the following:

- $(u[j], v[j]) \in D_{1}$ for all $j \in I^{\prime}$,
- $(u[m], v[m]) \in D_{2}$, and
- $(u[j], v[j]) \in D_{2}$ for all $j \in J^{\prime}$

By Observation 14 there exists $w \in A^{k}$ such that all of the following hold:

- $(w[j], u[j], v[j]) \in R_{1}$ for all $j \in I^{\prime}$,
- $(w[m], u[m], v[m]) \in\left\{\left(x_{1}, x_{2}, x_{3}\right) \in A^{3} \mid C_{1}\left(x_{1}, x_{2}\right) \wedge D_{2}\left(x_{2}, x_{3}\right) \wedge x_{1} \neq\right.$ $\left.x_{3}\right\}$, and
- $(w[j], u[j], v[j]) \in R_{2}$ for all $j \in J^{\prime}$.

By the induction hypothesis, it holds that $(f(w), f(u)) \in C_{1}$. Now, since $\mathbb{A}$ is liberal and $C_{1}, D_{1} \subseteq \neq$ there exist pairwise different $b_{1}, b_{2}, b_{3}$ such that $\left(b_{1}, b_{3}\right)$ are in the same orbital as $(w[m], v[m]),\left(b_{1}, b_{2}\right) \in C_{1}$ and $\left(b_{2}, b_{3}\right) \in D_{1}$. Since $\mathbb{A}$ is homogeneous, there is an automorphism $\alpha \in \mathbb{A}$ sending $\left(b_{1}, b_{3}\right)$ to $(w[m], v[m])$. Let $a$ be such that $a=\alpha\left(b_{2}\right)$ and $u^{\prime} \in A^{k}$ such that $u^{\prime}[m]=a$ and $u^{\prime}[j]=u[j]$ whenever $j \neq m$. Observe that $(w[m], a, v[m]) \in R_{1}$.

Since for all $j \in[k]$ we have that $\left(w[j], u^{\prime}[j], v[j]\right) \in R_{1}$ or $\left(w[j], u^{\prime}[j], v[j]\right) \in$ $R_{2}$, it follows that $\left(w[j], u^{\prime}[j], v[j]\right) \in R$ for all $j \in[k]$. The relation $R$ is preserved by $f$. Thus, $\left(f(w), f\left(u^{\prime}\right), f(v)\right) \in R$. The tuple $u$ is constant,
$f$ is an oligopotent qnu-operation, and hence $f\left(u^{\prime}\right)=f(u)$. It implies that $\left(f(w), f\left(u^{\prime}\right)\right) \in C_{1}$. The relation $R$ entails $\left(C_{1}\left(x_{1}, x_{2}\right) \Longrightarrow D_{1}\left(x_{2}, x_{3}\right)\right)$, and hence $\left(f\left(u^{\prime}\right), f(v)\right) \in D_{1}$. Since $f\left(u^{\prime}\right)=f(u)$, it follows that $(f(u), f(v)) \in D_{1}$. It contradicts the assumpution that $(f(u), f(v)) \notin D_{1}$ and proves that Item 2 holds for the induction step.

The proof that Item 1 goes through the induction step is analogous to the proof for Item 2 with a difference that we use Observation 13 instead of Observation 14. It completes the proof of the induction step and the observation.

Observation 16 implies $f\left(D_{2}, \ldots, D_{2}\right)=D_{1}$. It contradicts the fact that $D_{2}$ is pp-definable in $\mathbb{B}$ and completes the proof of the proposition.

## 4 Efficient Entailment and Implications

In this section, we first provide the definition of an implication, which is needed to formally define implicationally simple and implicationally hard structures. Then, in the following subsections, we prove certain preliminary results which are then used in the proof of the main theorem. In particular, in Section 4.1 we show some auxiliary results that will be used to pp-define critical ternary relations out of a pair of complementary implications. In Section 4.2 we show that some implications are not pp-definable in first-order expansions of liberal finitely bounded homogeneous binary cores with bounded strict width, in Section 4.3 how to compose implications and in Section 4.4 that out of a pair of complementary implications we can always pp-define an implication of a very concrete form called a complete implication.

We start with a definition of efficient entailment which is a non-standard version of entailment from Section 2 and concerns ternary and quaternary relations. First, we look into the ternary ones.

Definition 17 Let $R$ be a ternary relation and $C_{1}, D_{1}$ binary relations. We say that $R$ efficiently entails:

$$
\left(C_{1}\left(x_{i}, x_{j}\right) \Longrightarrow D_{1}\left(x_{k}, x_{l}\right)\right)
$$

with $i, j, k, l \in[3]$ if both of the following hold:

- $R\left(x_{1}, x_{2}, x_{3}\right)$ entails $\left(C_{1}\left(x_{i}, x_{j}\right) \Longrightarrow D_{1}\left(x_{k}, x_{l}\right)\right)$,
- $C_{1} \subsetneq \operatorname{proj}_{i, j} R$ and $D_{1} \subsetneq \operatorname{proj}_{k, l} R$.

We have a similar definition for quaternary relations.
Definition 18 Let $R$ be a quaternary relation and $C_{1}, D_{1}$ binary relations. We say that $R$ efficiently entails:

$$
\left(C_{1}\left(x_{i}, x_{j}\right) \Longrightarrow D_{1}\left(x_{k}, x_{l}\right)\right)
$$

with $i, j, k, l \in[4]$ if both of the following hold:

- $R\left(x_{1}, x_{2}, x_{3}, x_{4}\right)$ entails $\left(C_{1}\left(x_{i}, x_{j}\right) \Longrightarrow D_{1}\left(x_{k}, x_{l}\right)\right)$,
- $C_{1} \subsetneq \operatorname{proj}_{i, j} R$ and $D_{1} \subsetneq \operatorname{proj}_{k, l} R$.

Next we define a ternary 'implication'.
Definition 19 Let $L, P \in\{\leftarrow, \rightarrow\}$. We say that a ternary relation $R$ is a ternary $\left(\mathbb{B}, C, D, C_{1}, D_{1}, L, P\right)$-implication if all of the following hold:

1. $R$ entails no equalities,
2. all $R, C_{1}, D_{1}$ are pp-definable in $\mathbb{B}$,
3. $\operatorname{proj}_{i, j} R=C$ where $(i, j)=(1,2)$ if $L=\rightarrow$ and $(i, j)=(2,1)$ if $L=\leftarrow$,
4. $\operatorname{proj}_{k, l} R=D$ where $(k, l)=(2,3)$ if $L=\rightarrow$ and $(k, l)=(3,2)$ if $L=\leftarrow$,
5. Refficiently entails $\left(C_{1}\left(x_{i}, x_{j}\right) \Longrightarrow D_{1}\left(x_{k}, x_{l}\right)\right)$.

We now provide a similar definition for quaternary relations.
Definition 20 Let $L, P \in\{\leftarrow, \rightarrow\}$. A quaternary relation $R$ is a quaternary $\left(\mathbb{B}, C, D, C_{1}, D_{1}, L, P\right)$-implication if all of the following hold:

1. $R$ entails no equalities,
2. all $R, C_{1}, D_{1}$ are pp-definable in $\mathbb{B}$,
3. $\operatorname{proj}_{i, j} R=C$ where $(i, j)=(1,2)$ if $L=\rightarrow$ and $(i, j)=(2,1)$ if $L=\leftarrow$,
4. $\operatorname{proj}_{k, l} R=C$ where $(k, l)=(3,4)$ if $L=\rightarrow$ and $(k, l)=(4,3)$ if $L=\leftarrow$,
5. $R$ efficiently entails $\left(C_{1}\left(x_{i}, x_{j}\right) \Longrightarrow D_{1}\left(x_{k}, x_{l}\right)\right)$.

For the sake of succinctness we say that $R$ is a ( $\mathbb{B}, C, D, C_{1}, D_{1}, L, P$ )implication without specifying that it is ternary or quaternary if not necessary, or that a relation $R$ is a (ternary or quaternary) ( $\mathbb{B}, C, D, C_{1}, D_{1}$ )-implication if it is a ( $\mathbb{B}, C, D, C_{1}, D_{1}, L, P$ )-implication for some $L, P \in\{\leftarrow, \rightarrow\}$ or that $R$ is a (ternary or quaternary) $(L, P)$-implication if it is a $\left(\mathbb{B}, C, D, C_{1}, D_{1}, L, P\right)$ implication for some $\mathbb{B}, C, D, C_{1}, D_{1}$.

Example. Let the relation $R$ be a critical ternary relation over ( $\mathbb{B}, C_{1}, C_{2}$, $\left.D_{1}, D_{2}\right)$. Observe that the relation $R$ is a $\left(\mathbb{B}, \operatorname{proj}_{1,2} R, \operatorname{proj}_{2,3} R, C_{1}, D_{1}, \rightarrow, \rightarrow\right)$ implication and that

$$
R^{\prime}\left(x_{1}, x_{2}, x_{3}\right) \equiv R\left(x_{3}, x_{2}, x_{1}\right)
$$

is a $\left(\mathbb{B}, \operatorname{proj}_{3,2} R, \operatorname{proj}_{2,1} R, D_{1}^{-1}, C_{1}^{-1}, \rightarrow, \rightarrow\right)$-implication.

### 4.1 From Implications to Critical Ternary Relations

In order to define critical ternary relations, for instance, out of a pair (or a bunch) of implications we provide pp-definitions in many steps using often similar constructions. For the sake of succinctness, we will use certain shorthands.

Definition 21 We write $R_{3}:=R_{1} \bowtie R_{2}$ for a ternary relation $R_{3}\left(x_{1}, x_{2}, x_{3}\right)$ defined out of two ternary relations $R_{1}$ and $R_{2}$ as follows

$$
\begin{equation*}
\exists y R_{1}\left(x_{1}, x_{2}, y\right) \wedge R_{2}\left(y, x_{2}, x_{3}\right) \tag{1}
\end{equation*}
$$

or a quaternary relation $R_{3}\left(x_{1}, x_{2}, x_{3}, x_{4}\right)$ defined out of two quaternary relations $R_{1}, R_{2}$ as follows:

$$
\begin{equation*}
\exists y_{1} \exists y_{2} R_{1}\left(x_{1}, x_{2}, y_{1}, y_{2}\right) \wedge R_{2}\left(y_{2}, y_{1}, x_{3}, x_{4}\right) \tag{2}
\end{equation*}
$$

We also write $R_{1} \bowtie_{3} R_{2}$ for a ternary relation $R_{3}\left(x_{1}, x_{2}, x_{3}\right)$ defined out of two quaternary relations $R_{1}, R_{2}$ as follows:

$$
\begin{equation*}
\exists y_{1} \exists y_{2} R\left(x_{1}, x_{2}, y_{1}, y_{2}\right) \wedge R\left(y_{2}, y_{1}, x_{2}, x_{3}\right) \tag{3}
\end{equation*}
$$

The following observation provides us with some insight into the structure of $R_{1} \bowtie R_{2}$ on the condition that we provide some assumptions on $R_{1}, R_{2}$.

Observation 22 Let $R_{1}, R_{2}$ be both ternary or both quaternary relations fodefinable in a liberal finitely bounded homogeneous binary core $\mathbb{A}$, the relation $R_{3}:=R_{1} \bowtie R_{2}$ and $O_{1}, O_{2}, O_{3}$ some orbitals. Then all of the following hold:

- if $R_{1}$ contains an $O_{1} O_{2}$-tuple $t_{1}$ and $R_{2}$ contains an $O_{2}^{-1} O_{3}$-tuple $t_{2}$ then $R_{3}$ contains an $O_{1} O_{3}$-tuple,
- if $t_{1}$ and $t_{2}$ are injective, then $R_{3}$ contains all injective $O_{1} O_{3}$-tuples
- if $O_{1}, O_{2}, O_{3}$ are $=$ and both $t_{1}$ and $t_{2}$ are non-constant, then $R_{3}$ contains all non-constant $==$-tuples.

Proof: Consider the first item in the formulation of the observation and the case where $R_{1}$ and $R_{2}$ are ternary. Let $\left(a_{1}, a_{2}, a_{3}\right)$ be an $O_{1} O_{2}$-tuple in $R_{1}$ and $\left(b_{2}, b_{3}, b_{4}\right)$ an $O_{2}^{-1} O_{3}$-tuple in $R_{2}$. Since $\mathbb{A}$ is homogeneous, there exists an automorphism $\alpha \in \operatorname{Aut}(\mathbb{A})$ sending $\left(b_{3}, b_{2}\right)$ to $\left(a_{2}, a_{3}\right)$. Let $a_{4}$ be $\alpha\left(b_{4}\right)$. Observe that an assignment a : $\left\{x_{1}, x_{2}, x_{3}, y\right\} \rightarrow A$ such that $\mathrm{a}\left(x_{1}\right)=a_{1}, \mathrm{a}\left(x_{2}\right)=a_{2}$, $\mathrm{a}(y)=a_{3}$ and $\mathrm{a}\left(x_{3}\right)=a_{4}$. Since $\left(a_{3}, a_{2}, a_{4}\right)$ is in the same orbit as $\left(b_{2}, b_{3}, b_{4}\right)$, we have that the assignment a satisfies all atomic formulae in (1), we have that $R_{3}$ has an $O_{1} O_{3}$-tuple. The proof for quaternary $R_{1}, R_{2}$ is similar with a difference that we consider an $O_{1} O_{2}$-tuple $\left(a_{1}, a_{2}, a_{3}, a_{4}\right)$ in $R_{1}, O_{2}^{-1} O_{3}$-tuple $\left(b_{3}, b_{4}, b_{5}, b_{6}\right)$ in $R_{2}$, and an automorphism $\alpha \in \operatorname{Aut}(\mathbb{A})$ sending $\left(b_{4}, b_{3}\right)$ to $\left(a_{3}, a_{4}\right)$.

Now turn to the second item for ternary relation $R_{1}$ which contains an $O_{1} O_{2}{ }^{-}$ tuple $t_{1}$ and $R_{2}$ with an $O_{2}^{-1} O_{3}$-tuple $t_{2}$. Since $\mathbb{A}$ is liberal we have that for any orbital $O_{1,4}$ there exists a substructure of $\mathbb{A}$ over four pairwise different
elements $a_{1}, a_{2}, a_{3}, a_{4}$ such that $\left(a_{1}, a_{2}, a_{3}\right)$ is in the same orbit as $t_{1},\left(a_{3}, a_{2}, a_{4}\right)$ in the same orbit as $t_{2}$ and $\left(a_{1}, a_{4}\right) \in O_{1,4}$. It follows that $R_{3}$ has an $O_{1} O_{3^{-}}$ tuple $t$ such that $(t[1], t[3]) \in O_{1,4}$. Since $O_{1,4}$ was chosen arbitrarily, we have that $R_{3}$ contains all injective tuples. The proof for quaternary relations is again similar. We just look at a substructure of $\mathbb{A}$ over six elements $a_{1}, a_{2}, a_{3}, a_{4}, a_{5}, a_{6}$ such that $\left(a_{1}, a_{2}, a_{5}, a_{6}\right)$ is in the same orbit as $t_{1}$, the tuple $\left(a_{6}, a_{5}, a_{3}, a_{4}\right)$ is in the same orbit as $t_{2}$ and such that for all $(i, j) \in\{(1,3),(1,4),(2,3),(2,4)\}$ we have $\left(a_{i}, a_{j}\right) \in O_{i, j}$ for some anti-reflexive orbital $O_{i, j}$. An assignment a : $\left\{x_{1}, \ldots, x_{4}, y_{1}, y_{2}\right\} \rightarrow A$ such that $\mathrm{a}\left(x_{i}\right)=a_{i}$ for $i \in[4]$ and $\mathrm{a}\left(y_{i}\right)=a_{i+4}$ for $i \in[2]$ clearly satisfies both atomic formulae in (2). Since $O_{i, j}$ with $(i, j) \in$ $\{(1,3),(1,4),(2,3),(2,4)\}$ were chosen arbitrarily, it follows that $R_{3}$ contains all injective $O_{1} O_{3}$-tuples. It completes the proof of the second item.

Since there are no ternary non-constant $==$-tuples, the last item concerns quaternary relations only. We proceed as in the proof for the second item. Let $t_{1}$ be a non-constant tuple in $R_{1}$ and $t_{2}$ a non-constant tuple in $R_{2}$. This time we look at a substructure of $\mathbb{A}$ induced by pairwise different $a_{1}, a_{2}, a_{3}$ such that $\left(a_{1}, a_{2}\right)$ is in the same orbital as $\left(t_{1}[2], t_{1}[3]\right),\left(a_{2}, a_{3}\right)$ in the same orbit as $\left(t_{2}[2], t_{2}[3]\right)$, and $\left(a_{1}, a_{3}\right)$ in some anti-reflexive orbital $O_{1,3}$. The assignment a sending $x_{1}, x_{2}$ to $a_{1}, y_{1}, y_{2}$ to $a_{2}$ and $x_{3}, x_{4}$ to $a_{3}$ satisfies both atomic formulae in (2) and hence provides an $==$-tuple $t$ for $R$ satisfying $(t[2], t[3]) \in O_{1,3}$. Since $O_{1,3}$ was chosen arbitrarily, we have that $R_{3}$ contains all non-constant $==-$ tuples. It completes the proof for the last item in the observation.

An important part of the proof that a pp-defined ternary relation is a critical ternary relation is to show that it contains relations $\left(C_{1}\left(x_{1}, x_{2}\right) \wedge D_{1}\left(x_{2}, x_{3}\right)\right)$ and $\left(C_{2}\left(x_{1}, x_{2}\right) \wedge D_{2}\left(x_{2}, x_{3}\right)\right)$, see Definition 12. To this end we will use the following observation.

Observation 23 Let $R_{1}, R_{2}$ be two ternary relations with fo-definitions in a liberal finitely bounded homogeneous binary core $\mathbb{A}$ such that $R_{1}$ contains all injective $O_{1} O_{2}$-tuples and $R_{2}$ contains all injective $O_{2}^{-1} O_{3}$-tuples for some antireflexive orbitals $O_{1}, O_{2}, O_{3}$ then $R_{1} \bowtie R_{2}$ contains the relation

$$
O_{1}\left(x_{1}, x_{2}\right) \wedge O_{3}\left(x_{2}, x_{3}\right)
$$

Proof: If $O_{3}$ is different from $O_{1}^{-1}$, then the observation follows by Observation [22. Indeed, in this case it is enough to prove that $R_{3}$ contains all injective $O_{1} O_{3}$-tuples. Otherwise, we repeat the proof of Observation 22 to show that $R$ has all injective $O_{1} O_{1}^{-1}$-tuples but we have also to show that $R_{3}$ contains an $O_{1} O_{1}^{-1}$ tuple $t$ with $t[1]=t[3]$. To this end, consider any pairwise different $a_{1}, a_{2}, a_{3} \in A$ such that $\left(a_{1}, a_{2}\right) \in O_{1},\left(a_{2}, a_{3}\right) \in O_{2}$ and $\left(a_{1}, a_{3}\right) \in O_{1,3}$ for some anti-reflexive orbital $O_{1,3}$. They exist since $\mathbb{A}$ is liberal. The relation $R_{1}$ contains all injective $O_{1} O_{2}$-tuples and $R_{2}$ contains all injective $O_{2}^{-1} O_{1}^{-1}$-tuples, hence in particular $R_{1}$ contains $\left(a_{1}, a_{2}, a_{3}\right)$ and $R_{2}$ contains ( $a_{3}, a_{2}, a_{1}$ ). It follows that the assignment a: $\left\{x_{1}, x_{2}, x_{3}, y\right\}$ sending $x_{1}, x_{3}$ to $a_{1}, x_{2}$ to $a_{2}$ and $y$ to $a_{3}$ satisfies both atomic formulae in (1) and provides an $O_{1} O_{1}^{-1}$-tuple with $t[1]=t[3]$ in $R_{3}$.

The next observation will be used in a similar context as Observation 23 , The difference is that here we will look at quaternary not ternary relations $R_{1}, R_{2}$ and at $\bowtie_{3}$ not $\bowtie$.

Observation 24 Let $R_{1}, R_{2}$ be two quaternary relations with fo-definitions in a liberal finitely bounded homogeneous binary core $\mathbb{A}$ such that $R_{1}$ contains all injective $O_{1} O_{2}$-tuples and $R_{2}$ contains all injective $O_{2}^{-1} O_{3}$-tuples or all $O_{1}, O_{2}, O_{3}$ are $=$ and both $R_{1}, R_{2}$ contain all non-constant $==$-tuples then $R_{1} \bowtie_{3} R_{2}$ contains:

$$
O_{1}\left(x_{1}, x_{2}\right) \wedge O_{3}\left(x_{2}, x_{3}\right)
$$

Proof: Since $\mathbb{A}$ is liberal, we have that for anti-reflexive orbitals $O_{1,3}$ it contains a substructure over pairwise different elements $a_{1}, a_{2}, a_{3}, a_{4}, a_{5}$ such that $\left(a_{1}, a_{2}\right) \in O_{1},\left(a_{4}, a_{5}\right) \in O_{2},\left(a_{2}, a_{3}\right) \in O_{3}$ and $\left(a_{1}, a_{3}\right) \in O_{1,3}$. Since $R_{1}$ contains all injective $O_{1}, O_{2}$-tuples and $R_{2}$ contains all injective $O_{2}^{-1} O_{3}$-tuples, we have that an assignment a : $\left\{x_{1}, x_{2}, x_{3}, y_{1}, y_{2}\right\} \rightarrow A$ such that $\mathrm{a}\left(x_{i}\right)=a_{i}$ for $i \in[3]$ and $\mathrm{a}\left(y_{i}\right)=a_{i+3}$ for $i \in[2]$ satisfies both atomic formulae in (3). Since $O_{1,3}$ may be arbitrary, it proves that $R_{3}$ has all injective $O_{1} O_{3}$-tuples. If $O_{3}=O_{1}^{-1}$, then we also have to show that $R_{3}$ has an $O_{1} O_{1}^{-1}$-tuple $t$ with $t[1]=t[3]$. The proof is similar with a difference that we choose $a_{1}, a_{2}, a_{3}, a_{4}, a_{5}$ with $a_{1}=a_{3}$.

### 4.2 Some Implications with no Bounded Strict Width

We are now ready to prove that certain ternary and quaternary relations ppdefine a critical ternary relation and hence do not have bounded strict width.

Lemma 25 Let $\mathbb{A}$ be a liberal finitely bounded homogeneous binary core and $\mathbb{B}$ a first-order expansion of $\mathbb{A}$ which pp-defines a ternary relation $R$ that entails no equalities but entails

$$
\varphi\left(x_{1}, x_{2}, x_{3}\right) \equiv\left(x_{1}=x_{2} \vee x_{2}=x_{3} \vee x_{1}=x_{3}\right)
$$

Then $R$, and hence $\mathbb{B}$ do not have bounded strict width.

Proof: Since $R$ entails no equalities, we have that if $R$ entails any subformula of $\varphi$, then this subformula has at least two disjuncts. Assume without loss of generality that in this case $R$ entails ( $x_{1}=x_{2} \vee x_{2}=x_{3}$ ). Since $R$ entails no equalities we have that $\left(R^{\prime}\left(x_{1}, x_{2}, x_{3}\right) \equiv R\left(x_{1}, x_{2}, x_{3}\right) \wedge x_{1} \neq x_{3}\right)$ is equivalent to $\left(\left(C\left(x_{1}, x_{2}\right) \wedge x_{2}=x_{3}\right) \vee\left(x_{1}=x_{2} \wedge D\left(x_{2}, x_{3}\right)\right)\right)$ for some anti-reflexive $C, D \subseteq A^{2}$. By Observation 6, the relation $R^{\prime}$ is pp-definable in $\mathbb{B}$.

Further, since $\mathbb{A}$ is liberal, it is easy to see that the relation

$$
\begin{equation*}
\left(R^{\prime \prime}\left(x_{1}, x_{2}, x_{3}\right) \equiv \exists y R^{\prime}\left(x_{1}, x_{2}, y\right) \wedge R^{\prime}\left(x_{3}, x_{2}, y\right)\right) \tag{4}
\end{equation*}
$$

equals $\left(S\left(x_{1}, x_{2}, x_{3}\right) \equiv\left(C_{1}\left(x_{1}, x_{2}\right) \wedge C^{-1}\left(x_{2}, x_{3}\right)\right) \vee\left(x_{1}=x_{2} \wedge x_{2}=x_{3}\right)\right)$. Indeed, any assignment a : $\left\{x_{1}, x_{2}, x_{3}, y\right\} \rightarrow A$ satisfying both atomic formulae in (4)
either sends $\left(x_{1}, x_{2}\right)$ to some pair $\left(a_{1}, a_{2}\right)$ in $C_{1}$ or to the same element in $A$. In the former case, we have that $\mathrm{a}\left(x_{2}\right)=\mathrm{a}(y)$, and hence $\left(\mathrm{a}\left(x_{2}\right), \mathrm{a}\left(x_{3}\right)\right)=\left(a_{2}, a_{3}\right)$ for some $\left(a_{2}, a_{3}\right) \in C^{-1}$ Since $\mathbb{A}$ is liberal, we may choose $\left(a_{1}, a_{2}, a_{3}\right)$ so that $\left(a_{1}, a_{3}\right)$ is in any orbital $O_{1,3}$ if $\left(a_{1}, a_{2}\right) \in O_{1,2}$ and $\left(a_{2}, a_{3}\right)$ in $O_{2,3}=O_{1,2}^{-1}$ and so that $O_{1,3}$ is any anti-reflexive orbital if $O_{2,3} \neq O_{1,2}^{-1}$. It follows that $R^{\prime \prime}$ contains $\left(R_{1}\left(x_{1}, x_{2}, x_{3}\right) \equiv\left(C_{1}\left(x_{1}, x_{2}\right) \wedge C^{-1}\left(x_{2}, x_{3}\right)\right)\right)$ and that any tuple in $R^{\prime \prime}$ with the two first coordinates being different is in $R_{1}$. On the other hand, any assignment a : $\left\{x_{1}, x_{2}, x_{3}, y\right\} \rightarrow A$ satisfying both atomic formulae in (4) that sends $\left(x_{1}, x_{2}\right)$ to the same element in $A$, satisfies also $\left(\mathrm{a}\left(x_{2}\right), \mathrm{a}(y)\right) \in D$ and hence $\left(\mathrm{a}\left(x_{2}\right)=\mathrm{a}\left(x_{3}\right)\right)$. It follows that $R^{\prime \prime}$ equals $S$.

Clearly the relation $S$ efficiently entails both: $\left(C\left(x_{1}, x_{2}\right) \Longrightarrow C^{-1}\left(x_{2}, x_{3}\right)\right)$ and $\left(C^{-1}\left(x_{2}, x_{3}\right) \Longrightarrow C\left(x_{1}, x_{2}\right)\right)$ and clearly $C$ and $=$ are pp-definable in $\mathbb{B}$. Hence $R$ is a critical ternary relation over $\left(\mathbb{B}, C,=, C^{-1},=\right)$. It follows by Proposition 15 that $\mathbb{B}$ do not have bounded strict width.

Assume now that $R$ entails no subformulae of $\varphi$. Since $R$, however, entails $\varphi$, we have that also in this case $\left(R^{\prime}\left(x_{1}, x_{2}, x_{3}\right) \equiv R\left(x_{1}, x_{2}, x_{3}\right) \wedge x_{1} \neq x_{3}\right)$ is equivalent to $\left(\left(C\left(x_{1}, x_{2}\right) \wedge x_{2}=x_{3}\right) \vee\left(x_{1}=x_{2} \wedge D\left(x_{2}, x_{3}\right)\right)\right)$ for some antireflexive $C, D \subseteq A^{2}$. The proof is hence identical as in the previous case. It completes the proof of the lemma.

As we will see ( $\mathbb{B}, C, D, O,=$ )-implications where $\mathbb{B}$ is a first-order expansion of a liberal finitely bounded homogeneous binary core and $O$ an anti-reflexive orbital do not have bounded strict width. We start with ternary implications.

Lemma 26 Let $\mathbb{A}$ be a liberal finitely bounded homogeneous binary core and $\mathbb{B}$ a first-order expansion of $\mathbb{A}$, let $R$ be a ternary relation pp-definable in $\mathbb{B}$ that entails no equalities and efficiently entails

$$
\left(O\left(x_{1}, x_{2}\right) \Longrightarrow x_{2}=x_{3}\right)
$$

for some anti-reflexive orbital $O$. Then $\mathbb{B}$ does not have bounded strict-width.

Proof: Since $R$ entails no equalities, by Lemma 25, we have that $R$ contains an injective tuple $t$ such that in particular $(t[1], t[2]) \in P$ for some anti-reflexive orbital $P$ different from $O$. Consider now the relation:

$$
\begin{equation*}
R^{\prime}\left(x_{1}, x_{2}, x_{3}\right) \equiv \exists y R\left(x_{1}, x_{2}, y\right) \wedge O\left(x_{3}, y\right) \tag{5}
\end{equation*}
$$

and observe that $R^{\prime}$

- efficiently entails $\eta_{O}:=\left(O\left(x_{1}, x_{2}\right) \Longrightarrow O^{-1}\left(x_{2}, x_{3}\right)\right)$,
- contains an injective $O O^{-1}$-tuple, and
- an injective $P P^{-1}$ tuple.

The first item follows from the fact that the relation $R$ entails $\left(O\left(x_{1}, x_{2}\right) \Longrightarrow\right.$ $\left.x_{2}=x_{3}\right)$ ). For the second item consider a substructure of $\mathbb{A}$ induced by three pairwise different elements $a_{1}, a_{2}, a_{3}$ such that $\left(a_{1}, a_{2}\right) \in O$ and $\left(a_{2}, a_{3}\right) \in O^{-1}$.

Since $\mathbb{A}$ is liberal such $a_{1}, a_{2}, a_{3}$ exist. Observe that $\left(a_{1}, a_{2}, a_{3}\right)$ is an $O O^{-1}$-tuple in $R^{\prime}$. Finally, consider a substructure of $\mathbb{A}$ over four pairwise different elements $\left(a_{1}, a_{2}, a, a_{3}\right)$ such that $\left(a_{1}, a_{2}, a\right)$ is in the same orbit as $t,\left(a_{2}, a_{3}\right) \in P^{-1}$, and $\left(a_{3}, a\right) \in O$. Observe that an assignment sending $y$ to $a$ and $x_{i}$ to $a_{i}$ for $i \in[3]$ satisfies all atomic formulae in (5). It follows that $R$ contains an injective $P P^{-1}$-tuple.

Then by Observation 22 and 23, it follows that both the relation $R_{3}:=$ $\left(\left(R^{\prime} \bowtie R^{\prime}\right) \bowtie\left(R^{\prime} \bowtie R^{\prime}\right)\right)$ and the relation $\left(R_{3}^{\prime}\left(x_{1}, x_{2}, x_{3}\right) \equiv R_{3}\left(x_{3}, x_{2}, x_{1}\right)\right)$ contain both

- $R_{1}:=\left(O\left(x_{1}, x_{2}\right) \wedge O^{-1}\left(x_{2}, x_{3}\right)\right)$ and
- $R_{2}:=\left(P\left(x_{1}, x_{2}\right) \wedge P^{-1}\left(x_{2}, x_{3}\right)\right)$.

Clearly both $R_{3}$ and $R_{3}\left(x_{3}, x_{2}, x_{1}\right)$ efficiently entails $\eta_{O}$. It follows that the relation $\left(R_{4}\left(x_{1}, x_{2}, x_{3}\right) \equiv\left(R_{3}\left(x_{1}, x_{2}, x_{3}\right) \wedge R_{3}^{\prime}\left(x_{1}, x_{2}, x_{3}\right)\right)\right)$ is a critical ternary relation over ( $\mathbb{B}, O, P, O^{-1}, P^{-1}$ ) and the lemma follows by Proposition 15 .

Roughly speaking, the following corollary says that if a ternary relation $R$ contains an $O_{1} O_{2}$ tuple then it contains an injective $O_{1} P$-tuple for some orbital $P$ provided an orbital $O_{1}$ is anti-reflexive and an injective $O_{1} O_{2}$-tuple provided both orbitals $O_{1}$ and $O_{2}$ are anti-reflexive.

Corollary 27 Let $\mathbb{A}$ be a liberal finitely bounded homogeneous binary core and $\mathbb{B}$ a first-order expansion of $\mathbb{A}$ with bounded strict width that pp-defines a ternary relation $R$ that entails no equalities. Then for any list of pairwise different twoelement sets of indices $\left\{i_{1}, j_{1}\right\}, \ldots,\left\{i_{m}, j_{m}\right\}$ with $m \in[2]$ such that there exists a tuple $t \in R$ satisfying $t\left[i_{k}\right] \neq t\left[j_{k}\right]$ for all $k \in[m]$, the relation $R$ contains an injective tuple $t^{\prime}$ such that $\left(t^{\prime}\left[i_{k}\right], t^{\prime}\left[j_{k}\right]\right)$ is in the same orbital as $\left(t\left[i_{k}\right], t\left[j_{k}\right]\right)$ for all $k \in[m]$.

Proof: Assume the contary. Then, since $R$ entails no equalities, in the case where $m=1$ the relation $R$ efficiently entails $\left(O\left(x_{a}, x_{b}\right) \Longrightarrow x_{c}=x_{d}\right)$ for some $a, b, c, d \in[3]$. It contradicts Lemma 26 and completes the proof in this case.

For $m=2$ we have that $R$ entails the formula $\left(O\left(x_{a}, x_{b}\right) \wedge O\left(x_{b}, x_{c}\right) \Longrightarrow\right.$ $x_{a}=x_{c}$ ) where $\{a, b, c\}=\{1,2,3\}$. Without loss of generality assume that $R$ entails $\left(O\left(x_{1}, x_{2}\right) \wedge O\left(x_{1}, x_{3}\right) \Longrightarrow x_{2}=x_{3}\right)$. Observe that $R$ contains a tuple $t$ such that $(t[1], t[2]),(t[1], t[3]) \in O$ and $(t[2]=t[3])$. Since $R$ entails neither $\left(O\left(x_{1}, x_{2}\right) \Longrightarrow x_{2}=x_{3}\right)$ nor $\left(O\left(x_{1}, x_{3}\right) \Longrightarrow x_{2}=x_{3}\right)$. It follows that $R$ contains a tuple $t$ such that $(t[1], t[3]) \in O$ and $(t[2] \neq t[3])$. It follows that $\left(R^{\prime}\left(x_{1}, x_{2}, x_{3}\right) \equiv R\left(x_{1}, x_{2}, x_{3}\right) \wedge O\left(x_{1}, x_{3}\right)\right)$ efficiently entails $\left(O\left(x_{1}, x_{2}\right) \Longrightarrow\right.$ $\left.x_{2}=x_{3}\right)$. The corollary follows by Lemma 26 .

We now turn to showing that quaternary $(\mathbb{B}, C, D, O,=$ )-implications with anti-reflexive orbitals $O$ have no bounded strict width. We first consider the case where the implication has an injective tuple.

Lemma 28 Let $\mathbb{A}$ be a liberal finitely bounded homogeneous binary core and $\mathbb{B}$ a first-order expansion of $\mathbb{A}$ that pp-defines a quaternary relation $R$ that efficiently
entails

$$
\left(O\left(x_{1}, x_{2}\right) \Longrightarrow x_{3}=x_{4}\right)
$$

and contains an injective tuple. Then $\mathbb{B}$ does not have bounded strict width.

Proof: By the assumptions of the lemma, the relation $R$ contains an injective tuple $t_{P}$ such that $\left(t_{P}[1], t_{P}[2]\right) \in P$ for some anti-reflexive orbital $P$ different from $O$. By Corollary 27, the projection of the relation $R$ to its three first arguments contains an injective tuple $t^{\prime}$ with $\left(t^{\prime}[1], t^{\prime}[2]\right) \in O$, and hence $R$ contains an $O=$-tuple $t_{O}$ such that $t_{O}[1], t_{O}[2], t_{O}[3]$ are pairwise different. Further, we claim that the relation

$$
\begin{equation*}
R^{\prime}\left(x_{1}, x_{2}, x_{3}, x_{4}\right) \equiv \exists y R\left(x_{1}, x_{2}, x_{3}, y\right) \wedge O\left(x_{4}, y\right) \tag{6}
\end{equation*}
$$

satisfies all of the following:

- it efficiently entails $\left(O\left(x_{1}, x_{2}\right) \Longrightarrow O^{-1}\left(x_{3}, x_{4}\right)\right)$,
- contains an injective $O O^{-1}$-tuple, and
- an injective $P P^{-1}$-tuple.

The first item follows clearly by the fact that $R$ efficiently entails ( $\eta_{\mathrm{Eq}}:=$ $\left.\left(O\left(x_{1}, x_{2}\right) \Longrightarrow x_{3}=x_{4}\right)\right)$. For the second item, consider four pairwise different elements $a_{1}, a_{2}, a_{3}, a_{4}$ in $A$ such that $\left(a_{1}, a_{2}, a_{3}, a_{3}\right)$ is in the same orbit as $t_{O}$ and $\left(a_{3}, a_{4}\right) \in O^{-1}$. Since $\mathbb{A}$ is liberal the elements $a_{1}, a_{2}, a_{3}, a_{4}$ exist. Observe that the assignment a : $\left\{x_{1}, \ldots, x_{4}, y\right\} \rightarrow A$ sending $x_{i}$ to $a_{i}$ for $i \in[4]$ and $y$ to $a_{3}$ saisfies all atomic formulae in (6). It follows that $R^{\prime}$ has an injective $O O^{-1}$-tuple. For the last item consider pairwise different $a_{1}, a_{2}, a_{3}, a, a_{4} \in A$ such that $\left(a_{1}, a_{2}, a_{3}, a\right)$ is in the same orbit as $t_{P}$ and $a_{4}$ is such that $\left(a_{4}, a\right) \in O$ and $\left(a_{3}, a_{4}\right) \in P^{-1}$. Again, $a_{1}, a_{2}, a_{3}, a, a_{4}$ exist since $\mathbb{A}$ is liberal. Observe that the assignment a : $\left\{x_{1}, \ldots, x_{4}, y\right\} \rightarrow A$ sending $x_{i}$ to $a_{i}$ for $i \in[4]$ and $y$ to $a$ saisfies all atomic formulae in (6). Hence $R^{\prime}$ contains an injective $P P^{-1}$-tuple.

Then by Observation 22 and 24, we have that the ternary relation $R_{3}:=$ $\left(\left(R^{\prime} \bowtie R^{\prime}\right) \bowtie_{3}\left(R^{\prime} \bowtie R^{\prime}\right)\right)$ and the relation $\left(R_{3}^{\prime}\left(x_{1}, x_{2}, x_{3}\right) \equiv R_{3}\left(x_{3}, x_{2}, x_{1}\right)\right)$ contains both

- $R_{1}:=\left(O\left(x_{1}, x_{2}\right) \wedge O^{-1}\left(x_{2}, x_{3}\right)\right)$ and
- $R_{2}:=\left(P\left(x_{1}, x_{2}\right) \wedge P^{-1}\left(x_{2}, x_{3}\right)\right)$.

By the definitions of $\bowtie$ and $\bowtie_{3}$, both $R_{3}$ and $R_{3}^{\prime}$ entail $\left(\eta_{O} \equiv\left(O\left(x_{1}, x_{2}\right) \Longrightarrow\right.\right.$ $\left.O^{-1}\left(x_{2}, x_{3}\right)\right)$ ). Since both the relation $R_{3}$ and $R_{3}^{\prime}$ contain $R_{2}$, we have that both $R_{3}$ and $R_{3}^{\prime}$ efficiently entail $\eta_{O}$. It follows that $\left(R_{3}\left(x_{1}, x_{2}, x_{3}\right) \wedge R_{3}\left(x_{3}, x_{2}, x_{1}\right)\right)$ is a critical ternary relation over $\left(\mathbb{B}, O, P, O^{-1}, P^{-1}\right)$ and the lemma follows by Proposition 15

Next, we look at $(\mathbb{B}, C, D, O,=)$-implications $R$ where $O$ is an anti-reflexive orbital and $R$ does not have an injective tuple.

Lemma 29 Let $\mathbb{A}$ be a liberal finitely bounded homogeneous binary core and $\mathbb{B}$ a first-order expansion of $\mathbb{A}$ that pp-defines a quaternary relation $R$ that entails

$$
\left(x_{1}=x_{2} \vee x_{3}=x_{4}\right)
$$

and entails no equalities. Then $\mathbb{B}$ does not have bounded strict width.
Proof: Observe that every binary core with one injective orbital only is isomorphic to $(\mathbb{N} ; \neq=)$. By Theorem 1 in [22], see also the introduction in this paper, it follows that every relation $R$ that has both a first-order definition in $(\mathbb{N} ; \neq=)$ and bounded strict width is definable by a conjunction of disjunctions of disequalities, i.e., conjunctions of clauses of the form $\left(x_{1} \neq y_{1} \vee \cdots \vee x_{k} \neq y_{k}\right)$. Such relations $R$ clearly do not efficiently entail ( $x_{1}=x_{2} \vee x_{3}=x_{4}$ ), and hence we can assume that $\mathbb{A}$ contains at least two different anti-reflexive orbitals. This assumption will be in effect in the remainder of the proof of the lemma.

Let now $C, D \subseteq \neq$ be two anti-reflexive binary relations such that every $O=-$ tuple in $R$ with an antireflexive $O$ satisfies $O \in C$ and every $=O$-tuple in $R$ with an antireflexive $O$ satisfies $O \in D$. We set $O$ to be some orbital in $C$ and $P$ to be some anti-reflexive orbital different from $O$. We claim that the relation

$$
R^{\prime}\left(x_{1}, x_{2}, x_{3}, x_{4}\right) \equiv \exists y R\left(x_{1}, x_{2}, x_{3}, y\right) \wedge O\left(y, x_{4}\right)
$$

satisfies all of the following:

- $R^{\prime}$ efficiently entails $\left(P\left(x_{3}, x_{4}\right) \Longrightarrow x_{1}=x_{2}\right)$
- $R^{\prime}$ contains an injective tuple.

For the first item consider an assignment a : $\left\{x_{1}, x_{2}, x_{3}, x_{4}, y\right\}$ sending $\left(x_{3}, x_{4}\right)$ to $P$. Since $O$ is different from $P$, we have $\mathrm{a}\left(x_{3}\right) \neq \mathrm{a}(y)$. It implies $\mathrm{a}\left(x_{1}\right)=\mathrm{a}\left(x_{2}\right)$. Hence $R^{\prime}$ entails $\left(P\left(x_{3}, x_{4}\right) \Longrightarrow x_{1}=x_{2}\right)$. Further, consider $\left(a_{1}, a_{2}, a_{3}, a, a_{4}\right)$ such that $\left(a_{1}, a_{2}, a_{3}, a\right) \in R$ and $a_{3} \neq a,\left(a_{3}, a_{4}\right) \in P$, and $\left(a, a_{4}\right) \in O$. Since $\mathbb{A}$ is liberal and $R$ entails no equalities, such a tuple exists. It completes the proof of the first item.

For the second item consider $\left(a_{1}, a_{2}, a_{3}, a, a_{4}\right)$ such that $\left(a_{1}, a_{2}, a_{3}, a\right) \in R$, $a_{3}=a$, and $\left(a_{3}, a_{4}\right) \in O$, and $\left(a_{1}, a_{2}\right) \in O^{\prime}$ for some anti-reflexive orbital $O^{\prime}$. We will show that $\left(a_{1}, a_{2}, a_{3}, a_{4}\right)$ is an injective $O^{\prime} O$-tuple in $R^{\prime}$ or $R$ has no bounded strict width. If the former is not the case, then $R^{\prime}$ entails $\left(O^{\prime}\left(x_{1}, x_{2}\right) \wedge O\left(x_{3}, x_{4}\right) \Longrightarrow x_{k}=x_{l}\right)$ for some $k, l \in[4]^{2} \backslash\{\{1,2\},\{3,4\}\}$. Since $R$ entails no equalities, $\{k, l\} \cap\{1,2\} \neq \emptyset$ and $\{k, l\} \cap\{3,4\} \neq \emptyset$, we have by Lemma 26 that $R^{\prime}$ either has no bounded strict width and we are done or entails neither $O^{\prime}\left(x_{1}, x_{2}\right) \Longrightarrow x_{k}=x_{l}$ nor $O\left(x_{3}, x_{4}\right) \Longrightarrow x_{k}=x_{l}$. It follows that $\left(R^{\prime \prime}\left(x_{1}, x_{2}, x_{3}, x_{4}\right) \wedge O\left(x_{3}, x_{4}\right)\right)$ efficiently entails $\left(O^{\prime}\left(x_{1}, x_{2}\right) \Longrightarrow x_{k}=x_{l}\right)$. It follows by Lemma 26 that $R$ has no bounded strict width and completes the proof of the second item.

Now, it is easy to see that Lemma 28 applied to $\mathbb{A}, \mathbb{B}$ and the relation $\left(R^{\prime \prime}\left(x_{1}, x_{2}, x_{3}, x_{4}\right) \equiv R^{\prime}\left(x_{3}, x_{4}, x_{1}, x_{2}\right)\right)$ completes the proof of this lemma.

Practically, the following corollary says that for any non-constant tuple $t$ of a quaternary relation under consideration one can find an injective tuple $t^{\prime}$ that agrees with $t$ on the 'injective part'.

Corollary 30 Let $\mathbb{A}$ be a liberal finitely bounded homogeneous binary core and $\mathbb{B}$ a first-order expansion of $\mathbb{A}$ with bounded strict width that pp-defines a quaternary relation $R$ that entails no equalities. Then for any list of pairwise different two-element sets of indices $\left\{i_{1}, j_{1}\right\}, \ldots,\left\{i_{m}, j_{m}\right\}$ with $i_{k}, j_{k} \in[4]$ for all $k \in[m]$ such that there exists a tuple $t \in R$ satisfying $t\left[i_{k}\right] \neq t\left[j_{k}\right]$ for all $k \in[m]$, the relation contains an injective tuple $t^{\prime}$ such that $\left(t^{\prime}\left[i_{k}\right], t^{\prime}\left[j_{k}\right]\right)$ is in the same orbital as $\left(t\left[i_{k}\right], t\left[j_{k}\right]\right)$ for all $k \in[m]$.

Proof: Assume the contrary and let $O_{i_{k}, j_{k}}$ with $k \in[m]$ be an orbital of $t\left[i_{k}, j_{k}\right]$. Then there exists a minimal subset $\mathbb{I} \subseteq\left\{\left\{i_{1}, j_{1}\right\}, \ldots\left\{i_{m}, j_{m}\right\}\right\}$ such that the formula

$$
\left(R\left(x_{1}, x_{2}, x_{3}, x_{4}\right) \wedge \bigwedge_{\{i, j\} \in \mathbb{I}} O_{i, j}\left(x_{j}, x_{j}\right)\right)
$$

entails $y_{1}=y_{2}$ for some $y_{1}, y_{2} \in\left\{x_{1}, x_{2}, x_{3}, x_{4}\right\}$. Assume without loss of generality that $\left(y_{1}, y_{2}\right)=\left(x_{3}, x_{4}\right)$. Let now $\mathbb{I}^{\prime}:=\mathbb{I} \backslash\left\{i_{0}, j_{0}\right\}$ for some $\left\{i_{0}, j_{0}\right\}$ in II. Since $R$ entails no equalities, the pair $\left\{i_{0}, j_{0}\right\}$ exists. Observe that the new formula

$$
\left(R^{\prime}\left(x_{1}, x_{2}, x_{3}, x_{4}\right) \equiv R\left(x_{1}, x_{2}, x_{3}, x_{4}\right) \wedge \bigwedge_{\{i, j\} \in \mathbb{I}^{\prime}} O_{i, j}\left(x_{i}, x_{j}\right)\right)
$$

efficiently entails $\left(O_{i_{0}, j_{0}}\left(x_{i_{0}}, x_{j_{0}}\right) \Longrightarrow x_{3}=x_{4}\right)$. If there is such a choice of $\mathbb{I}^{\prime}$ and $i_{0}, j_{0}$ that $\left\{i_{0}, j_{0}\right\} \cap\{3,4\} \neq \emptyset$, then the corollary follows by Lemma 26. Otherwise $\left\{i_{0}, j_{0}\right\}=\{1,2\}$ and $R^{\prime}$ contains a tuple $t$ such that $t_{1}[1], t_{1}[2], t_{1}[3]$ are pairwise different and $t_{1}[3]=t_{1}[4]$ as well as a tuple $t_{2}$ such that $t_{2}[2], t_{2}[3], t_{1}[4]$ are pairwise different. If we can find $t_{2}$ which is injective, then we are done by Lemma 28. Otherwise $R^{\prime}$ entails $\left(x_{1}=x_{2} \vee x_{3}=x_{4}\right)$. Since $R^{\prime}$ entails no equalities, the corollary follows by Lemma 29.

### 4.3 Composing Implications

We will now define the way the 'implications' can be composed. We will be composing the implications originating from a 'cyclic' instance of $\operatorname{CSP}(\mathbb{B})$ for implicationally hard $\mathbb{B}$ in order to obtain a critical ternary relation.

Definition 31 Let the relation $R_{1}$ be a $\left(\mathbb{B}, C, D, C_{1}, D_{1}, L_{1}, P_{1}\right)$-implication and $R_{2}$ a $\left(\mathbb{B}, D, F, D_{1}, F_{1}, L_{2}, P_{2}\right)$-implication. We write $R_{3}:=R_{1} \circ R_{2}$ for a relation obtained in one of the following ways:

- if both $R_{1}$ and $R_{2}$ are quaternary implications and $P_{1}=L_{2}$, then $R_{3}$ is quaternary and $R_{3}\left(x_{1}, x_{2}, x_{3}, x_{4}\right)$ is defined by:

$$
\begin{equation*}
\exists y_{1} \exists y_{2} R_{1}\left(x_{1}, x_{2}, y_{1}, y_{2}\right) \wedge R_{2}\left(y_{1}, y_{2}, x_{3}, x_{4}\right) \tag{7}
\end{equation*}
$$

- if both $R_{1}$ and $R_{2}$ are quaternary implications and $P_{1} \neq L_{2}$, then $R_{3}$ is quaternary and $R_{3}\left(x_{1}, x_{2}, x_{3}, x_{4}\right)$ is defined by:

$$
\begin{equation*}
\exists y_{1} \exists y_{2} R_{1}\left(x_{1}, x_{2}, y_{1}, y_{2}\right) \wedge R_{2}\left(y_{2}, y_{1}, x_{3}, x_{4}\right) \tag{8}
\end{equation*}
$$

- if the relation $R_{1}$ is a quaternary $\left(L_{1}, P_{1}\right)$-implication and the relation $R_{2}$ a ternary $\left(L_{2}, P_{2}\right)$-implication such that $P_{1}=L_{2}$, then $R_{3}$ is quaternary and $R_{3}\left(x_{1}, x_{2}, x_{3}, x_{4}\right)$ is defined by

$$
\begin{equation*}
\exists y R_{1}\left(x_{1}, x_{2}, y, x_{3}\right) \wedge R_{2}\left(y, x_{3}, x_{4}\right) \tag{9}
\end{equation*}
$$

- if the relation $R_{1}$ is a quaternary $\left(L_{1}, P_{1}\right)$-implication and the relation $R_{2}$ a ternary $\left(L_{2}, P_{2}\right)$-implication such that $P_{1} \neq L_{2}$, then $R_{3}$ is quaternary and $R_{3}\left(x_{1}, x_{2}, x_{3}, x_{4}\right)$ is defined by

$$
\begin{equation*}
\exists y R_{1}\left(x_{1}, x_{2}, x_{3}, y\right) \wedge R_{2}\left(y, x_{3}, x_{4}\right) \tag{10}
\end{equation*}
$$

- if the relation $R_{1}$ is a ternary $\left(L_{1}, P_{1}\right)$-implication and $R_{2}$ is a quaternary $\left(L_{2}, P_{2}\right)$-implication such that $\left(P_{1}=L_{2}\right)$, then $R_{3}$ is quaternary and $R_{3}\left(x_{1}, x_{2}, x_{3}, x_{4}\right)$ is defined by

$$
\begin{equation*}
\exists y R_{1}\left(x_{1}, x_{2}, y\right) \wedge R_{2}\left(x_{2}, y, x_{3}, x_{4}\right) \tag{11}
\end{equation*}
$$

- if the relation $R_{1}$ is a ternary $\left(L_{1}, P_{1}\right)$-implication and $R_{2}$ is a quaternary $\left(L_{2}, P_{2}\right)$-implication such that $\left(P_{1} \neq L_{2}\right)$, then $R_{3}$ is quaternary and $R_{3}\left(x_{1}, x_{2}, x_{3}, x_{4}\right)$ is defined by

$$
\begin{equation*}
\exists y R_{1}\left(x_{1}, x_{2}, y\right) \wedge R_{2}\left(y, x_{2}, x_{3}, x_{4}\right) \tag{12}
\end{equation*}
$$

- if the relation $R_{1}$ is a ternary $\left(L_{1}, P_{1}\right)$-implication and the relation $R_{2}$ a ternary $\left(L_{2}, P_{2}\right)$-implication with $P_{1}=L_{1}$, then $R_{3}$ is quaternary and $R_{3}\left(x_{1}, x_{2}, x_{3}, x_{4}\right)$ is defined by

$$
\begin{equation*}
R_{1}\left(x_{1}, x_{2}, x_{3}\right) \wedge R_{2}\left(x_{2}, x_{3}, x_{4}\right) \tag{13}
\end{equation*}
$$

- if the relation $R_{1}$ is a ternary $\left(L_{1}, P_{1}\right)$-implication and the relation $R_{2}$ a ternary $\left(L_{2}, P_{2}\right)$-implication with $P_{1} \neq L_{2}$, then $R_{3}$ is ternary and $R_{3}\left(x_{1}, x_{2}, x_{3}, x_{4}\right)$ is defined by

$$
\begin{equation*}
\exists y R_{1}\left(x_{1}, x_{2}, y\right) \wedge R_{2}\left(y, x_{2}, x_{3}\right) \tag{14}
\end{equation*}
$$

We will also write $\left(R_{1}\right)^{\circ k}$ for $(\underbrace{R_{1} \circ \cdots \circ R_{1}}_{k \text { times }})$.
We will now prove that o states for the composition of implications, i.e., that this operation also returns an implication.

Lemma 32 Let the relation $R_{1}$ be a $\left(\mathbb{B}, C, D, C_{1}, D_{1}, L_{1}, P_{1}\right)$-implication and $R_{2}$ a $\left(\mathbb{B}, D, F, D_{1}, F_{1}, L_{2}, P_{2}\right)$-implication such that both have a first-order definition in a liberal finitely bounded homogeneous structure $\mathbb{A}$. Then the relation $R_{3}:=R_{1} \circ R_{2}$ is a $\left(\mathbb{B}, C, F, C_{1}, F_{1}, L_{1}, P_{2}\right)$-implication such that $R_{3}$ has an $O_{1} O_{3}$-tuple for some orbitals $O_{1}$ and $O_{3}$ if

- either $P_{1}=L_{2}, R_{1}$ has an $O_{1} O_{2}$-tuple and $R_{2}$ has an $O_{2} O_{3}$-tuple for some orbital $O_{2}$, or
- $P_{1} \neq L_{2}, R_{1}$ has an $O_{1} O_{2}$-tuple and $R_{2}$ has an $O_{2}^{-1} O_{3}$-tuple for some orbital $O_{2}$.

Proof: Consider first the case where both $R_{1}, R_{2}$ are quaternary implications such that $L_{2}=P_{1}$ and let $\left(a_{1}, a_{2}, a_{3}, a_{4}\right)$ and $\left(b_{3}, b_{4}, b_{5}, b_{6}\right)$ be an $O_{1} O_{2}$-tuple in $R_{1}$ and an $O_{2} O_{3}$-tuple in $R_{2}$ for some orbitals $O_{1}, O_{2}, O_{3}$, respectively. Since $\mathbb{A}$ is homogeneous, we have that there exists an automorphism $\alpha$ of $\mathbb{A}$ sending $\left(a_{3}, a_{4}\right)$ to $\left(b_{3}, b_{4}\right)$. Let $\left(b_{1}, b_{2}\right)=\left(\alpha\left(a_{1}\right), \alpha\left(a_{2}\right)\right)$. It is now easy to see that an assignment $\mathrm{a}\left(x_{1}\right)=b_{1}, \mathrm{a}\left(x_{2}\right)=b_{2}, \mathrm{a}\left(y_{1}\right)=b_{3}, \mathrm{a}\left(y_{2}\right)=b_{4}, \mathrm{a}\left(x_{3}\right)=b_{5}, \mathrm{a}\left(x_{4}\right)=b_{6}$ satisfies both atomic formulae in (7) and since $\left(b_{5}, b_{6}\right) \in O_{3}$, it provides the desired $O_{1} O_{3 \text { - }}$ tuple in $R_{3}$. Since $\operatorname{proj}_{3,4} R_{1}=\operatorname{proj}_{1,2} R_{2}$, we have that $\operatorname{proj}_{1,2} R_{3}=\operatorname{proj}_{1,2} R_{1}$ and $\operatorname{proj}_{3,4} R_{2}=\operatorname{proj}_{3,4} R_{3}$. Further, $R_{1}\left(x_{1}, x_{2}, y_{1}, y_{2}\right)$ entails $\left(C_{1}\left(x_{i}, x_{j}\right) \Longrightarrow\right.$ $\left.D_{1}\left(y_{k}, y_{l}\right)\right)$ and $R_{2}\left(y_{1}, y_{2}, x_{3}, x_{4}\right)$ entails $\left(D_{1}\left(y_{k}, y_{l}\right) \Longrightarrow F_{1}\left(x_{m}, x_{n}\right)\right)$ where $i, j, k, l \in[2]$ and $m, n \in\{3,4\}$. It follows that $R_{3}$ entails $\left(\eta\left(x_{1}, x_{2}, x_{3}, x_{4}\right) \equiv\right.$ $\left.\left(C_{1}\left(x_{i}, x_{j}\right) \Longrightarrow F_{1}\left(x_{k}, x_{l}\right)\right)\right)$. Since $C_{1} \subsetneq \operatorname{proj}_{1,2} R_{3}=\operatorname{proj}_{1,2} R_{1}$ and $F_{1} \subsetneq$ $\operatorname{proj}_{3,4} R_{3}=\operatorname{proj}_{3,4} R_{2}$, it follows that $\eta$ is efficiently entailed by $R_{3}$. In order to complete the proof for this case, it is enough to prove that $R_{3}$ entails no equalities. Indeed, since $R_{1}$ entails no equalities, it contains an $O_{1} O_{2}$-tuple where $O_{1}$ is anti-reflexive. By Corollary 30 it follows that $R_{3}$ contains an injective tuple $t^{\prime}$ with $\left(t^{\prime}[1], t^{\prime}[2]\right) \in O_{1}$. It implies that $R_{3}$ implies no equalities, and hence that $R_{3}:=R_{1} \circ R_{2}$ is a $\left(\mathbb{B}, C, F, C_{1}, F_{1}, L_{1}, P_{2}\right)$-implication.

The proof in the case where both $R_{1}$ and $R_{2}$ are quaternary implications but $L_{2} \neq P_{1}$ is similar with a difference that we look at an $O_{1} O_{2}$-tuple ( $a_{1}, a_{2}, a_{3}, a_{4}$ ) and an $O_{2}^{-1} O_{3}$-tuple ( $b_{3}, b_{4}, b_{5}, b_{6}$ ) and search for an automorphism $\alpha$ that sends $\left(a_{3}, a_{4}\right)$ to $\left(b_{4}, b_{3}\right)$.

Further, the proof in all other cases in analogous to either of the cases considered above. In the last case where $R_{3}$ is a ternary implication, we use Corollary 27 instead of 30 .

### 4.4 Bipartite Digraph of Implications

In this subsection we introduce a bipartite graph that reflects the structure of $O P$-tuples inside a pair of ternary or quaternary implications. Observe that the relations $\left(C_{i}\left(x_{1}, x_{2}\right) \wedge D_{i}\left(x_{2}, x_{3}\right)\right)$ with $i \in[2]$ definable in liberal finitely bounded homogeneous binary cores, which are important ingredients of critical ternary relations, have all possible $O P$-tuples such that $O \subseteq C_{i}$ and $P \subseteq D_{i}$.

Consider a finitely bounded homogeneous structure $\mathbb{A}$ and $C \subseteq A^{2}$. We will write $\operatorname{Vert}_{L}(C)$ and $\operatorname{Vert}_{R}(C)$ for the set $\left\{O_{L} \mid O\right.$ is an orbital contained in $\left.C\right\}$
and the set $\left\{O_{R} \mid O\right.$ is an orbital contained in $\left.C\right\}$, respectively. To keep things simple, we say that a $\left(\mathbb{B}, C, C, C_{1}, C_{1}, L, P\right)$-implication is a $\left(\mathbb{B}, C, C_{1}, L, P\right)$ implication.

Definition 33 Let the relation $R_{1}$ and the relation $R_{2}$ be both $\left(\mathbb{B}, C, C_{1}, \rightarrow\right.$, $\leftarrow)$-implications. We define a bipartite directed graph $\mathcal{B}_{R_{1}, R_{2}}$ over two disjoint sets of vertices $\operatorname{Vert}_{L}(C)$ and $\operatorname{Vert}_{R}(C)$. The digraph $\mathcal{B}_{R_{1}, R_{2}}$ contains

- an arc $\left(O_{L}, P_{R}\right) \in \operatorname{Vert}_{L}(C) \times \operatorname{Vert}_{R}(C)$ if the relation $R_{1}$ contains a tuple $\left(a_{1}, a_{2}, a_{3}, a_{4}\right)$ such that $\left(a_{1}, a_{2}\right) \in O$ and $\left(a_{3}, a_{4}\right) \in P^{-1}$ and
- an arc $\left(P_{R}, O_{L}\right) \in \operatorname{Vert}_{R}(C) \times \operatorname{Vert}_{L}(C)$ if the relation $R_{2}$ contains a tuple $\left(a_{1}, a_{2}, a_{3}, a_{4}\right)$ such that $\left(a_{1}, a_{2}\right) \in P$ and $\left(a_{3}, a_{4}\right) \in O^{-1}$.

We say that $\left\{O_{L}, P_{R}\right\}$ is a symmetric edge in $\mathcal{B}_{R_{1}, R_{2}}$ if it contains both an $\operatorname{arc}\left(O_{L}, P_{R}\right)$ and $\left(P_{R}, O_{L}\right)$.

We say that a subset $S$ of the vertices of a digraph is a strongly connected component if it is a maximal set of vertices such that for any two vertices $u, v \in S$ there is a path connecting $u$ and $v$. A set of vertices $S$ is a set of strongly connected components if it can be partitioned into $S_{1}, \ldots, S_{k}$ so that every $S_{i}$ with $i \in[k]$ is a strongly connected component. We say that a (set of) strongly connected components is a sink if every arc originating in $S$ finishes in $S$ and that $S$ is a source if every arc finishing in $S$ also originates in $S$.

Lemma 34 Let $R_{1}, R_{2}$ be $\left(\mathbb{B}, C, C_{1}, \rightarrow, \leftarrow\right)$-implications, $\mathbb{B}$ a first-order expansion of a liberal finitely bounded homogeneous binary core $\mathbb{A}$. Then $\left(\operatorname{Vert}_{L}\left(C_{1}\right) \cup\right.$ $\left.\operatorname{Vert}_{R}\left(C_{1}\right)\right)$ is a set of strongly connected components which is a sink in $\mathcal{B}_{R_{1}, R_{2}}$.

Moreover, there exist non-empty $D_{1}, F_{1} \subseteq C \backslash C_{1}$ such that $\left(\operatorname{Vert}_{L}\left(D_{1}\right) \cup\right.$ $\left.\operatorname{Vert}_{R}\left(F_{1}\right)\right)$ is a source in $\mathcal{B}_{R_{1}, R_{2}}$.

Proof: For the first part of the lemma observe that every arc originating in some vertex in $\left(\operatorname{Vert}_{L}\left(C_{1}\right) \cup \operatorname{Vert}_{R}\left(C_{1}\right)\right)$ finishes in a vertex in $\left(\operatorname{Vert}_{L}\left(C_{1}\right) \cup\right.$ $\left.\operatorname{Vert}_{R}\left(C_{1}\right)\right)$. Indeed, it follows by the fact that both $R_{1}, R_{2}$ are $\left(\mathbb{B}, C, C_{1}, \rightarrow, \leftarrow\right.$ )-implications.

Observe that the second part of the lemma follows by the facts that the component $\left(\operatorname{Vert}_{L}\left(C_{1}\right) \cup \operatorname{Vert}_{R}\left(C_{1}\right)\right)$ is a sink in $\mathcal{B}_{R_{1}, R_{2}}$, the fact that $C_{1} \subsetneq C$ and that $\mathcal{B}_{R_{1}, R_{2}}$ is finite and smooth, i.e., has no sources and no sinks. The graph $\mathcal{B}_{R_{1}, R_{2}}$ is smooth since $\operatorname{proj}_{1,2} R_{1}=\operatorname{proj}_{1,2} R_{2}=\operatorname{proj}_{4,3} R_{1}=\operatorname{proj}_{4,3} R_{2}$.

The following lemma may be simply proved by induction using Lemma 32
Lemma 35 Let $R_{1}, R_{2}$ be $\left(\mathbb{B}, C, C_{1}, \rightarrow, \leftarrow\right)$-implications, $\mathbb{B}$ a first-order expansion of a liberal finitely bounded homogeneous binary core $\mathbb{A}$ and $O_{L}, P_{R}$ a pair of vertices in $\mathcal{B}_{R_{1}, R_{2}}$ such that there is a path from $O_{L}$ to $P_{R}$ of length $2 k+1$ for some $k \in \mathbb{N}$. Then $\left(R_{1} \circ R_{2}\right)^{\circ k} \circ R_{1}$ is a $\left(\mathbb{B}, C, C_{1}, \rightarrow, \leftarrow\right)$-implication and has an $O P^{-1}$-tuple.

We will say that the relation $R$ is a complete $\left(\mathbb{B}, C, C_{1}, \rightarrow, \leftarrow\right)$-implication if every strongly connected component in $\mathcal{B}_{R, R}$ satisfies both of the following:

- it is of the form $\operatorname{Vert}_{L}\left(D_{1}\right) \cup \operatorname{Vert}_{R}\left(D_{1}\right)$ for some $D_{1} \subseteq C$,
- every strongly connected component of $\mathcal{B}_{R, R}$ is a complete bipartite digraph.

Example. Let $\mathbb{B}$ be a first-order expansion of a liberal finitely bounded homogeneous binary core and $R$ a critical ternary relation over $\left(\mathbb{B}, C_{1}, D_{1}, C_{1}^{-1}, D_{1}^{-1}\right)$ given by

$$
\left(C_{1}\left(x_{1}, x_{2}\right) \wedge C_{1}^{-1}\left(x_{2}, x_{3}\right)\right) \vee\left(D_{1}\left(x_{1}, x_{2}\right) \wedge D_{1}^{-1}\left(x_{2}, x_{3}\right)\right)
$$

Observe that the relation $R$ is a complete $\left(\mathbb{B}, C_{1} \cup D_{1}, C_{1}, \rightarrow, \leftarrow\right)$-implication.
We finally prove that we can always define a complete implication given two appropriate implications.

Lemma 36 Let $R_{1}, R_{2}$ be both $\left(\mathbb{B}, C, C_{1}, \rightarrow, \leftarrow\right)$-implications where $\mathbb{B}$ is a firstorder expansion of a liberal finitely bounded homogeneous binary core $\mathbb{A}$. Then they pp-define a complete $\left(\mathbb{B}, C, C_{1}, \rightarrow, \leftarrow\right)$-implication.

Proof: We say that two vertices in a bipartite digraph are loosely connected if they are in the same strongly connected component of the digraph and the shortest cycle they are both involved in is of length strictly greater than 2.

Consider now two ( $\mathbb{B}, C, C_{1}, \rightarrow, \leftarrow$ )-implications $R_{1}, R_{2}$ such that not every strongly connected component of $\mathcal{B}_{R_{1}, R_{2}}$ is complete. Then there are loosely connected vertices $O_{L}, P_{R}$ in $\mathcal{B}_{R_{1}, R_{1}}$ such that there is an $\operatorname{arc}\left(P_{R}, O_{L}\right)$ in $\mathcal{B}_{R_{1}, R_{2}}$ but the shortest path from $O_{L}$ to $P_{R}$ is of length $2 k+1$ with $k \geq 1$. By Lemma 35, we have that $R_{3}:=\left(R_{1} \circ R_{2}\right)^{\circ k} \circ R_{1}$ contains an $O P^{-1}$-tuple as well as all $O^{\prime} P^{\prime-1}$-tuples such that $\left\{O^{\prime}, P^{\prime}\right\}$ is a symmetric edge in $\mathcal{B}_{R_{1}, R_{2}}$. Since $R_{3}$ is also a $\left(\mathbb{B}, C, C_{1}, \rightarrow, \leftarrow\right)$-implication, we have that the number of loosely connected vertices in $\mathcal{B}_{R_{3}, R_{2}}$ is strictly less than in $\mathcal{B}_{R_{1}, R_{2}}$. Since the graphs under consideration are finite, it is enough to repeat the whole procedure a finite number of times in order to obtain a pair of implications $R^{\prime}, R^{\prime \prime}$ without losely connected vertices.

In order to complete the proof of the lemma observe that $R^{\prime} \circ R^{\prime \prime}$ is the desired complete $\left(\mathbb{B}, C, C_{1}, \rightarrow, \leftarrow\right)$-implication.

## 5 Implicationally Simple Languages

In this section we characterize the relational width of a large class of constraint languages which we call implicationally simple. We start with a precise definition of $\mathcal{G}_{\mathcal{I}}$.

Graph of a CSP-instance. Let $\mathcal{I}$ be a $\left(2, \mathbb{L}_{\mathbb{A}}\right)$-minimal instance of $\operatorname{CSP}(\mathbb{B})$ over variables $\mathcal{V}$ of which we assume that $\mathcal{I}$ entails no equalities. We define the implication graph $\mathcal{G}_{\mathcal{I}}$ of $\mathcal{I}$ to be a directed graph over vertices which are triples of the form $\left(\left(v_{1}, v_{2}\right), C\right)$ where $v_{1}, v_{2} \in \mathcal{V}$ and $C \subsetneq \mathcal{I}_{v_{1}, v_{2}}$ is a binary non-empty relation pp-definable in $\mathbb{B}$. There is an arc in one of the two following situations:

- an $\left.\operatorname{arc}\left(\left(x_{1}, x_{2}\right), C_{1}\right),\left(\left(x_{2}, x_{3}\right), D_{1}\right)\right)$ with pairwise different $x_{1}, x_{2}, x_{3} \in \mathcal{V}$ if there exists a constraint $\mathbf{C} \in \mathcal{I}$ whose scope contains $\left\{x_{1}, x_{2}, x_{3}\right\}$,
$-\operatorname{proj}_{x_{1}, x_{2}, x_{3}} \mathbf{C}=\left(\left(x_{1}, x_{2}, x_{3}\right), R^{\prime}\right)$ and
- $R^{\prime}$ is a ternary $\left(\mathbb{B}, \mathcal{I}_{x_{1}, x_{2}}, C_{1}, \mathcal{I}_{x_{2}, x_{3}}, D_{1}\right)$-implication,
- an $\left.\operatorname{arc}\left(\left(x_{1}, x_{2}\right), C_{1}\right),\left(\left(x_{3}, x_{4}\right), D_{1}\right)\right)$ with paiwise different $x_{1}, x_{2}, x_{3}, x_{4} \in$ $\mathcal{V}$ if there exists a constraint $\mathbf{C} \in \mathcal{I}$ whose scope contains $\left\{x_{1}, x_{2}, x_{3}, x_{4}\right\}$ and
$-\operatorname{proj}_{x_{1}, x_{2}, x_{3}, x_{4}} \mathbf{C}=\left(\left(x_{1}, x_{2}, x_{3}, x_{4}\right), R^{\prime}\right)$ and
- the relation $R^{\prime}$ is a quaternary $\left(\mathbb{B}, \mathcal{I}_{x_{1}, x_{2}}, C_{1}, \mathcal{I}_{x_{3}, x_{4}}, D_{1}\right)$-implication.

We are now ready to define a new class of languages.
Definition 37 Let $\mathbb{B}$ be a first-order expansion of a finitely bounded homogeneous binary core $\mathbb{A}$. We say that a structure $\mathbb{B}$ is implicationally simple if for every $\left(2, \mathbb{L}_{\mathbb{A}}\right)$ instance $\mathcal{I}$ of $\operatorname{CSP}(\mathbb{B})$ the graph $\mathcal{G}_{\mathcal{I}}$ is acyclic.

If $\mathbb{B}$ is not implicationally simple, then we say that it is implicationally hard.
Naturally, every acyclic $\mathcal{G}_{\mathcal{I}}$ contains a sink which is a singleton $\left\{\left(\left(v_{1}, v_{2}\right), C\right\}\right)$ for some $v_{1}, v_{2} \in \mathcal{V}$ and $C \subsetneq A^{2}$. We will now show that in this case an instance $\mathcal{I}\left[\left(v_{1}, v_{2}\right):=C\right]$ obtained from $\mathcal{I}$ by narrowing down the projection of every constraint in $\mathcal{I}$ containing $\left(v_{1}, v_{2}\right)$ in its scope to $C$ has the good properties of $\mathcal{I}$.

Observation 38 Let $\mathcal{I}$ be a non-trivial $\left(2, \mathbb{L}_{\mathbb{A}}\right)$-minimal instance of $\operatorname{CSP}(\mathbb{B})$ where $\mathbb{B}$ is a first-order expansion of a finitely bounded homogeneous binary core $\mathbb{A}$ such that $\left\{\left(\left(v_{1}, v_{2}\right), C\right)\right\}$ is a sink in $\mathcal{G}_{\mathcal{I}}$. Then the instance $\mathcal{I}\left[\left(v_{1}, v_{2}\right):=C\right]$ is also a non-trivial $\left(2, \mathbb{L}_{\mathbb{A}}\right)$-minimal instance of $\operatorname{CSP}(\mathbb{B})$.

Proof: Since $\mathcal{I}$ is $\left(2, \mathbb{L}_{\mathbb{A}}\right)$-minimal, we have that $\mathcal{I}\left[\left(v_{1}, v_{2}\right):=C\right]$ is nontrivial. To see that $\mathcal{I}\left[\left(v_{1}, v_{2}\right):=C\right]$ is $\left(2, \mathbb{L}_{\mathbb{A}}\right)$-minimal consider any constraint $\mathbf{C}=\left(\left(x_{1}, \ldots, x_{r}\right), R\right)$ in $\mathcal{I}$ and any pair of variables $v_{3}, v_{4} \subseteq\left\{x_{1}, \ldots, x_{r}\right\}$. We have to show that for every orbital $O \subseteq \mathcal{I}_{v_{3}, v_{4}}$ there is an assignment a : $\left\{x_{1}, \ldots, x_{r}\right\} \rightarrow A$ satisfying $R\left(x_{1}, \ldots, x_{r}\right)$ such that $\left(\mathrm{a}\left(v_{3}\right), \mathrm{a}\left(v_{4}\right)\right) \in O$. If $\left\{v_{1}, v_{2}\right\} \nsubseteq\left\{x_{1}, \ldots, x_{r}\right\}$, then we are done by the fact that $\mathcal{I}$ is $\left(2, \mathbb{L}_{\mathbb{A}}\right)$ minimal. In the other case, since $\left\{\left(\left(v_{1}, v_{2}\right), C\right)\right\}$ is a sink in $\mathcal{G}_{\mathcal{I}}$, the projection of $R\left(x_{1}, \ldots, x_{r}\right)$ to $\left\{v_{1}, v_{2}, v_{3}, v_{4}\right\}$ does not efficiently entail $\left(C\left(v_{1}, v_{2}\right) \rightarrow D\left(v_{3}, v_{4}\right)\right)$ for any $D \subsetneq I_{v_{3}, v_{4}}$. Hence there exists an assignment a : $\left\{x_{1}, \ldots, x_{r}\right\} \rightarrow A$ satisfying $R\left(x_{1}, \ldots, x_{r}\right)$ such that $\left(\mathrm{a}\left(v_{3}\right), \mathrm{a}\left(v_{4}\right)\right) \in O$ and $\left(\mathrm{a}\left(v_{1}\right), \mathrm{a}\left(v_{2}\right)\right) \in C$. It completes the proof of the observation.

The above observation is the key to showing that every non-trivial $\left(2, \mathbb{L}_{\mathbb{A}}\right)$ minimal instance of $\operatorname{CSP}(\mathbb{B})$ with implicationally simple $\mathbb{B}$ has a solution.

Theorem 39 Let $\mathbb{A}$ be a finitely-bounded homogenous binary core and $\mathbb{B}$ a firstorder expansion of $\mathbb{A}$ which is implicationally simple. Then every non-trivial $\left(2, \mathbb{L}_{\mathbb{A}}\right)$-minimal instance has a solution.

Proof: Indeed, either there exist $v_{1}, v_{2} \in \mathcal{V}$ such that $I_{v_{1}, v_{2}}$ consists of at least two orbitals or all $I_{v_{1}, v_{2}}$ consist of exactly one orbital $O$. In the former case, since all binary relations pp-definable in $\mathbb{B}$ contain an orbital, the graph $\mathcal{G}_{\mathcal{I}}$ contains at least one vertex $\left(\left(v_{1}, v_{2}\right), C\right)$ for $v_{1}, v_{2} \in \mathcal{V}$ such that $\mathcal{I}_{v_{1}, v_{2}}$ consists of at least two orbits. Since $\mathbb{B}$ is implicationally simple, the graph $\mathcal{G}_{\mathcal{I}}$ is acyclic. In particular, it has a sink $\left\{\left(\left(v_{1}, v_{2}\right), C\right)\right\}$ for some $C$. Hence, we can use Observation 38 and simplify the considered instance by replacing $\mathcal{I}$ with $\mathcal{I}\left[\left(v_{1}, v_{2}\right):=C\right]$. The new instance is also non-trivial and $\left(2, \mathbb{L}_{\mathbb{A}}\right)$-minimal. The process of simplifying the instance terminates when every $\mathcal{I}_{v_{1}, v_{2}}$ with $v_{1}, v_{2} \in \mathcal{V}$ consists of one orbit only. Clearly, every solution to this simplified instance is a solution to the original one.

From now we assume that $\mathcal{I}$ is $\left(2, \mathbb{L}_{\mathbb{A}}\right)$-minimal, $\mathcal{V}=\left\{v_{1}, \ldots, v_{n}\right\}$ and for all $i, j \in[n]$ we have that $\mathcal{I}_{v_{i}, v_{j}}$ consists of exactly one orbital. Consider a structure $\Delta$ over the signature $\tau$ of $\mathbb{A}$ whose elements are variables in $\mathcal{V}$ and $\left(v_{i}, v_{j}\right) \in R^{\Delta}$ with $R \in \tau$ if and only if $\mathcal{I}_{v_{i}, v_{j}}=R^{\mathbb{A}}$. Observe first that $E$ equal to $=^{\Delta} \cup\left\{\left(v_{i}, v_{i}\right) \mid i \in[n]\right\}$ is an equivalence relation. It is clearly reflexive and symmetric. Suppose $E$ is not transitive. Then there are $v_{i}, v_{j}, v_{k} \in V$ such that $\left(v_{i}, v_{j}\right) \in=^{\Delta}$ and $\left(v_{j}, v_{k}\right) \in=^{\Delta}$ but $v_{i}, v_{k} \notin=^{\Delta}$. It implies that a constraint $\mathbf{C}$ whose scope contains $\left\{v_{i}, v_{j}, v_{k}\right\}$, which is in $\mathcal{I}$ by $\left(2, \mathbb{L}_{\mathbb{A}}\right)$-minimality, has an empty relation. It contradicts the fact that $\mathcal{I}$ is non-trivial.

In order to complete the proof of the theorem we will show that $\Delta / E$ admits an embedding to $\mathbb{A}$. Assume it is not the case. Then there exists a finite structure $\Gamma \in \mathcal{F}_{\mathbb{A}}$ that embeds into $\mathbb{A}$. Since the size of $\Gamma$ is at most $\mathbb{L}_{\mathbb{A}}$, it contradicts the fact that $\mathcal{I}$ is $\left(2, \mathbb{L}_{\mathbb{A}}\right)$-minimal and non-trivial. It follows that $\Delta / E$ embeds and $\Delta$ homomorphically maps into $\mathbb{A}$. It follows that $\mathcal{I}$ has a solution.

## 6 FO-Expansions of Liberal Finitely Bounded Homogeneous Binary Cores without Bounded Strict Width

In this section we show that if a first-order expansion of a liberal finitely bounded homogeneous binary core is implicationally hard, then it has no bounded strict width. As a result, we obtain that every first-order expansion of a liberal finitely bounded homogeneous binary core with bounded strict width is implicationally simple and hence its relational width is characterized by Theorem 39.

Lemma 40 Let $\mathbb{A}$ be a liberal finitely bounded homogeneous binary core and $\mathbb{B}$ a first-order expansion of $\mathbb{A}$ which is implicationally hard. Then $\mathbb{B}$ pp-defines

- a quaternary $\left(\mathbb{B}, C, C_{1}, \rightarrow, \leftarrow\right)$-implication, or
- a ternary $\left(\mathbb{B}, C, C_{1}, \rightarrow, \leftarrow\right)$-implication.

Proof: Since $\mathbb{B}$ is implicationally hard, there exists an instance $\mathcal{I}$ of $\operatorname{CSP}(\mathbb{B})$ such that $\mathcal{G}_{\mathcal{I}}$ contains a path $\left((v, z), C_{1}\right)=\left(\left(v_{1}, z_{1}\right), D_{1}\right), \ldots,\left(\left(v_{n}, z_{n}\right), D_{n}\right)=$ $\left((v, z), C_{1}\right)$ such that for all $j \in[n-1]$ the instance $\mathcal{I}$ contains a constraint $\mathbf{C}$ whose scope contains $v_{j}, z_{j}, v_{j+1}, z_{j+1}$ and one of the following holds:

- either the elements $x_{1}=v_{j}, x_{2}=z_{j}=v_{j+1}, x_{3}=z_{j+1}$ are pairwise different, $\operatorname{proj}_{x_{1}, x_{2}, x_{3}} \mathbf{C}=\left(\left(x_{1}, x_{2}, x_{3}\right), R_{j}^{\prime}\right)$ and the relation $R_{j}^{\prime}$ is a ternary $\left(\mathbb{B}, \mathcal{I}_{v_{j}, z_{j}}, \mathcal{I}_{v_{j+1}, z_{j+1}}, C_{j}, C_{j+1}\right)$-implication, or
- the elements $x_{1}=v_{j}, x_{2}=z_{j}, x_{3}=v_{j+1}, x_{4}=z_{j+1}$ are pairwise different, $\operatorname{proj}_{x_{1}, x_{2}, x_{3}, x_{4}} \mathbf{C}=\left(\left(x_{1}, x_{2}, x_{3}, x_{4}\right), R_{j}^{\prime}\right)$ and $R_{j}^{\prime}$ is a quaternary $\left(\mathbb{B}, \mathcal{I}_{v_{j}, z_{j}}, \mathcal{I}_{v_{j+1}, z_{j+1}}, C_{j}, C_{j+1}\right)$-implication.
It follows that $R^{\prime}:=\left(\left(R_{1}^{\prime} \circ R_{2}^{\prime}\right) \circ \ldots \circ R_{k}^{\prime}\right)$ is well defined and by Lemma 32, the relation $R^{\prime}$ is a $\left(\mathbb{B}, \mathcal{I}_{v, z}, \mathcal{I}_{v, z}, C_{1}, C_{1}\right)$-implication. Assume first that $R^{\prime}$ is ternary. If the relation $R^{\prime}$ is a ternary $\left(\mathbb{B}, \mathcal{I}_{v, z}, C_{1}, \leftarrow, \rightarrow\right)$-implication, then it is also a ternary $\left(\mathbb{B},\left(\mathcal{I}_{v, z}\right)^{-1},\left(C_{1}\right)^{-1}, \rightarrow, \leftarrow\right)$-implication and then the lemma follows. Otherwise, $R^{\prime}$ is a $(\rightarrow, \rightarrow)$-implication or a $(\leftarrow, \leftarrow)$-implication, and then, by Lemma40, the relation $R^{\prime} \circ R^{\prime}$ is a quaternary $\left(\mathbb{B}, \mathcal{I}_{v, z}, C_{1}\right)$-implication. From now on we may therefore assume that $R^{\prime}$ is quaternary.

If the relation $R^{\prime}$ is a $(\rightarrow, \leftarrow)$-implication, then we are done. Otherwise $R^{\prime}$ is an $(L, P)$-quaternary implication with either $L \neq \rightarrow$ or $P \neq \leftarrow$. But then

- $R^{\prime}\left(x_{2}, x_{1}, x_{3}, x_{4}\right)$ is a $(\rightarrow, \leftarrow)$-implication in the case where $L=\leftarrow$ and $R=\leftarrow$,
- $R^{\prime}\left(x_{1}, x_{2}, x_{4}, x_{3}\right)$ is a $(\rightarrow, \leftarrow)$-implication in the case where $L=\rightarrow$ and $R=\rightarrow$, and
- $R^{\prime}\left(x_{2}, x_{1}, x_{4}, x_{3}\right)$ is a $(\rightarrow, \leftarrow)$-implication in the case where $L=\leftarrow$ and $R=\rightarrow$.

It completes the proof of the lemma.
We are now in the position to show that first-order expansions of liberal finitely bounded homogenous binary cores $\mathbb{A}$ with bounded strict width are implicationally simple and hence their relational width is $\left(2, \mathbb{L}_{\mathbb{A}}\right)$.

Lemma 41 Let $\mathbb{A}$ be a liberal finitely bounded homogeneous binary core and $\mathbb{B}$ a first-order expansion of $\mathbb{A}$ which is implicationally hard. Then $\mathbb{B}$ has no bounded strict width.

Proof: By Lemma 40, we may assume that $\mathbb{B}$ pp-defines an $\left(\mathbb{B}, C^{\prime}, C_{1}^{\prime}, \rightarrow\right.$ $, \leftarrow)$-implication $R^{\prime}$. By Lemma 36 that $R^{\prime}$ is a complete implication. By Lemma 34, we have that $\left(\operatorname{Vert}_{L}\left(C_{1}^{\prime}\right) \cup \operatorname{Vert}_{R}\left(C_{1}^{\prime}\right)\right)$ is a set of strongly connected components which is a sink in $\mathcal{B}_{R^{\prime}, R^{\prime}}$. Let $C_{1}^{\prime \prime} \subseteq A^{2}$ be such that $\left(\operatorname{Vert}_{L}\left(C_{1}^{\prime \prime}\right) \cup\right.$ $\left.\operatorname{Vert}_{R}\left(C_{1}^{\prime \prime}\right)\right)$ form a strongly connected component included in $\left(\operatorname{Vert}_{L}\left(C_{1}^{\prime}\right) \cup\right.$ $\left.\operatorname{Vert}_{R}\left(C_{1}^{\prime}\right)\right)$ which is a sink in $\mathcal{B}_{R^{\prime}, R^{\prime}}$ and $D_{1}^{\prime \prime} \subseteq A^{2}$ such that $\left(\operatorname{Vert}_{L}\left(D_{1}^{\prime \prime}\right) \cup\right.$ $\left.\operatorname{Vert}_{R}\left(D_{1}^{\prime \prime}\right)\right)$ is a strongly connected component which is a source in $\mathcal{B}_{R^{\prime}, R^{\prime}}$. Since the component $\left(\operatorname{Vert}_{L}\left(C_{1}^{\prime \prime}\right) \cup \operatorname{Vert}_{R}\left(C_{1}^{\prime \prime}\right)\right)$ is a sink, we have that $R^{\prime}$ is a $\left(\mathbb{B}, C^{\prime}, C_{1}^{\prime \prime}, \rightarrow, \leftarrow\right)$-implication, Now, if either $C_{1}^{\prime}$ or $D_{1}^{\prime}$ is neither anti-reflexive nor it is $=$, then we replace $R^{\prime}$ with $R^{\prime \prime}:=R^{\prime}\left(x_{1}, x_{2}, x_{3}\right) \wedge x_{1} \neq x_{2} \wedge x_{2} \neq$ $x_{3}$ if $R^{\prime}$ is ternary and with $R^{\prime \prime}:=R^{\prime}\left(x_{1}, x_{2}, x_{3}, x_{4}\right) \wedge x_{1} \neq x_{2} \wedge x_{3} \neq x_{4}$ if $R^{\prime}$ is quaternary. Set now $C=C^{\prime} \backslash\{=\}, C_{1}=C_{1}^{\prime \prime} \cap C$ and $D_{1}=D_{1}^{\prime \prime} \cap$ $C$ and observe that $\mathcal{B}_{R^{\prime \prime}, R^{\prime \prime}}$ is a digraph induced in $\mathcal{B}_{R^{\prime}, R^{\prime}}$ by $\left(\operatorname{Vert}_{L}(C) \cup\right.$ $\operatorname{Vert}_{R}(C)$ ). It follows that $R^{\prime \prime}$ is a complete $\left(\mathbb{B}, C, C_{1}, \rightarrow, \leftarrow\right)$-implication where the strongly connected component $\left(\operatorname{Vert}_{L}\left(C_{1}\right) \cup \operatorname{Vert}_{R}\left(C_{1}\right)\right)$ is a sink and the strongly connected component $\left(\operatorname{Vert}_{L}\left(D_{1}\right) \cup \operatorname{Vert}_{R}\left(D_{1}\right)\right)$ is a source in $\mathcal{B}_{R^{\prime \prime}, R^{\prime \prime}}$.

From now on we will assume that the relation $R$ is a complete $\left(\mathbb{B}, C, C_{1}, \rightarrow\right.$ $, \leftarrow)$-implication where $C_{1}$ is either $=$ or it is anti-reflexive and $\left(\operatorname{Vert}_{L}\left(C_{1}\right) \cup\right.$ $\left.\operatorname{Vert}_{R}\left(C_{1}\right)\right)$ is a strongly connected component which is a sink in $\mathcal{B}_{R, R}$ as well as that $D_{1} \subseteq C \backslash C_{1}$ is either $=$ or it is anti-reflexive and $\left(\operatorname{Vert}_{L}\left(D_{1}\right) \cup \operatorname{Vert}_{R}\left(D_{1}\right)\right)$ is a strongly connected component which is a source in $\mathcal{B}_{R, R}$. Clearly, either $C_{1}$ or $D_{1}$ is anti-reflexive. We will now prove that $R$ pp-defines a ternary $\left(\mathbb{B}, C, C_{1}, \rightarrow, \leftarrow\right)$-implication $R_{1}$ that contains for all $F_{1} \subseteq\left\{C_{1}, D_{1}\right\}$ the relation $\left(F_{1}\left(x_{1}, x_{2}\right) \wedge F_{1}^{-1}\left(x_{2}, x_{3}\right)\right)$. In the case where $R$ is ternary we set $R_{1}:=(R \bowtie$ $R) \bowtie(R \bowtie R)$ which is equivalent to setting $R_{1}:=(R \circ R) \circ(R \circ R)$. Hence, by Lemma 32, the relation $R$ is a ternary $\left(\mathbb{B}, C, C_{1}, \rightarrow, \leftarrow\right)$-implication. Further, if $F_{1}$ is $=$, then clearly $R_{1}$ has a ternary $==-$ tuple. If $F_{1}$ is anti-reflexive, then consider any orbitals $O_{1}, O_{3}$ in $F_{1}$. Since $R$ is a complete implication, by Corollary 27 it contains an injective $O_{1} O_{2}^{-1}$-tuple as well as an injective $\mathrm{O}_{2} \mathrm{O}_{3}^{-1}$-tuple for some orbital $\mathrm{O}_{2} \subseteq F_{1}$, and hence by Observations 22 and 23 , we have that $R_{1}$ contains $\left(O_{1}\left(x_{1}, x_{2}\right) \wedge O_{3}^{-1}\left(x_{2}, x_{3}\right)\right)$, and in consequence that $R_{1}$ contains $\left(F_{1}\left(x_{1}, x_{2}\right) \wedge F_{1}^{-1}\left(x_{2}, x_{3}\right)\right)$. Clearly, also the relation $\left(\left(R_{2}\left(x_{1}, x_{2}, x_{3}\right) \equiv\right.\right.$ $\left.R_{1}\left(x_{1}, x_{2}, x_{3}\right) \cap R_{1}\left(x_{3}, x_{2}, x_{1}\right)\right)$ ) contains $\left(F_{1}\left(x_{1}, x_{2}\right) \wedge F_{1}^{-1}\left(x_{2}, x_{3}\right)\right)$ for $F_{1} \subseteq$ $\left\{C_{1}, D_{1}\right\}$. Further, observe that $\mathcal{B}_{R_{2}, R_{2}}$ has only symmetric edges. Hence, since $\left(\operatorname{Vert}_{L}\left(D_{1}\right) \cup \operatorname{Vert}_{R}\left(D_{1}\right)\right)$ is a sink in $\mathcal{B}_{R, R}$ and also in $\mathcal{B}_{R_{2}, R_{2}}$, we have that $D_{1}\left(x_{1}, x_{2}\right) \equiv R_{2}\left(x_{1}, x_{2}, x_{3}\right) \wedge O^{-1}\left(x_{2}, x_{3}\right)$ for some orbital contained in $D_{1}$, and hence $D_{1}$ is pp-definable in $\mathbb{B}$. It follows that $R_{2}$ is either a critical ternary relation over $\left(\mathbb{B}, C_{1}, D_{1}, C_{1}^{-1}, D_{1}^{-1}\right)$ if $C_{1}$ is anti-reflexive or a critical ternary relation over $\left(\mathbb{B}, D_{1}, C_{1}, D_{1}^{-1}, C_{1}^{-1}\right)$ if $D_{1}$ is anti-reflexive. By appeal to Proposition 15 the structure $\mathbb{B}$ does not have bounded strict width.

We now turn to the case where $R$ is a complete quaternary $\left(\mathbb{B}, C, C_{1}, \rightarrow, \leftarrow\right)$ implication. This time we look at $R_{1}:=\left((R \bowtie R) \bowtie_{3}(R \bowtie R)\right)$ and this time we will first show that for all $F_{1} \in\left\{C_{1}, D_{1}\right\}, R_{1}$ contains the relation $R_{F}:=F_{1}\left(x_{1}, x_{2}\right) \wedge F_{1}^{-1}\left(x_{3}, x_{4}\right)$. If $F_{1}$ is $=$ and $R$ contains a constant tuple, then $R_{1}$ also contains a constant tuple and hence $R_{F}$. If $F_{1}$ is $=$ and $R_{1}$ contains a non-constant $==$-tuple, then by Observations 22 and 24, $R_{1}$ contains
an $==$-tuple and hence $R_{F}$. For the case where $F_{1}$ is anti-reflexive we again consider any orbitals $O_{1}, O_{3}^{-1}$ in $F$. Since $R$ is a complete implication, it contains an injective $O_{1} O_{2}^{-1}$-tuple as well as an injective $O_{2} O_{3}^{-1}$-tuple for some some orbital $O_{2} \subseteq F_{1}$. It follows by Observations 22 and 24 that $R_{1}$ contains $R_{F}$ also in the case where $F_{1}$ is anti-reflexive. Consider now $\left(R_{2}\left(x_{1}, x_{2}, x_{3}\right) \equiv\right.$ $\left.R_{1}\left(x_{1}, x_{2}, x_{3}\right) \cap R_{1}\left(x_{3}, x_{2}, x_{1}\right)\right)$ and obeserve that $\mathcal{B}_{R_{2}, R_{2}}$ has only symmetric edges. Since $\left(\operatorname{Vert}_{L}\left(D_{1}\right) \cup \operatorname{Vert}_{R}\left(D_{1}\right)\right)$ is a strongly connected component in $\mathcal{B}_{R_{2}, R_{2}}$ we can pp-define $D_{1}$. Hence $R_{2}$ is either a critical ternary relation over $\left(\mathbb{B}, C_{1}, D_{1}, C_{1}^{-1}, D_{1}^{-1}\right)$ if $C_{1}$ is anti-reflexive or a critical ternary relation over $\left(\mathbb{B}, D_{1}, C_{1}, D_{1}^{-1}, C_{1}^{-1}\right)$ if $D_{1}$ is anti-reflexive. By Proposition 15 we have that $\mathbb{B}$ does not have bounded strict width.
We are now ready to provide the proof for Theorem 1 .
Proof of Theorem 1 The result follows by Lemma 41 and Theorem 39

## 7 Future Work

In this paper we characterized the relational width of first order expansions of liberal finitely bounded homogenous binary cores with bounded strict width. First of all it is natural to ask what happens if the binary core is not liberal. By Proposition 3 we have that the method in this paper does not work when we forbid structures of size 3. On the other hand, by the result in [23] it holds that the relational width of first-order expansions of homogenous graphs $\mathbb{A}$ even with $\mathbb{L}_{\mathbb{A}}=3$ is still $\left(2, \mathbb{L}_{\mathbb{A}}\right)$. Could it be true for all binary cores $\mathbb{A}$ ?

Last but not least, what happens to the relational width when we allow first-order expansions of cores $\mathbb{A}$ of arbitrary arity $k$ ? We believe that with an appropriate notion of liberal $k$-ary cores $\mathbb{A}$ one can show that their first-order expansions $\mathbb{B}$ with bounded strict width have relational width $\left(k, \mathbb{L}_{\mathbb{A}}\right)$. Could it be true for all reducts of finitely bounded homogeneous structures?

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